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REGULARITY IN OVERLAPPING GENERATIONS EXCHANGE ECONOMIES*

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In this paper we develop a regularity theory for stationary overlapping generations economies. We show that generically there is an odd number of steady states in which a non-zero amount of nominal debt (fiat money) is passed from generation to generation and an odd number in which there is no nominal debt. We are also interested in non-steady state perfect foresight paths. As a first step in this direction we analyze the behavior of paths near a steady state. We show that generically they are given by a second order difference equation that satisfies strong regularity properties. Economic theory alone imposes little restriction on these paths: With n goods and consumers who live for m periods, for example, the only restriction on the set of paths converging to the steady state is that they form a manifold of dimension no less than one, no more than $2nm$.

1. Introduction

The theory of regularity developed by Debreu (1970) for static exchange economics has played an important role in recent studies of the comparative statics properties of general equilibrium models. In this paper we develop a regularity theory for stationary overlapping generation exchange economies.

We begin by studying steady states. We show that generically there is an odd number of steady states in which a non-zero amount of nominal debt (fiat money) is passed from generation to generation and an odd number in which there is no nominal debt. Generically, these latter steady states have price levels that tend either to zero or to infinity. We are also interested in non-steady state perfect foresight paths. As a first step in this direction we analyze the behavior of paths near a steady state. We show that generically

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they are given by a difference equation that satisfies strong regularity properties. Economic theory alone imposes little restriction on these paths: with n goods and consumers who live m periods, for example, the only restriction on the set of paths converging to the steady state is that they form a manifold of dimension no less than one, no more than $2nm$.

The regularity theory we develop here can be applied to analyze the response of an overlapping generations economy to unanticipated shocks. Kehoe and Levine (1982a) consider the impact of shocks under alternative assumptions about the types of contractual arrangements existing before the shock and the process by which perfect foresight forecasts are formed.

2. The model

We analyze a stationary overlapping generations model that generalizes that introduced by Samuelson (1958). Initially we assume that consumers live two periods. The model with many periods of life is discussed in section 11. In each period there are n goods. Each generation t is identical and consumes in periods t and $t+1$. The consumption and savings decisions of the (possibly many different types of) consumers in generation t are aggregated into excess demand functions $y(p_t, p_{t+1})$ in period t and $z(p_t, p_{t+1})$ in period $t+1$. The vector $p_t = (p_t^1, \ldots, p_t^n)$ denotes the prices prevailing in period t. Excess demand is assumed to satisfy the following assumptions:

A.1. (*Differentiability*) $y, z: R^{2n}_{++} \to R^n$ are smooth (that is, C^1) functions.
A.2. (*Walras's law*) $p_t' y(p_t, p_{t+1}) + p_{t+1}' z(p_t, p_{t+1}) \equiv 0$.
A.3. (*Homogeneity*) y, z are homogeneous of degree zero.
A.4. (*Boundary*) $\|(y(q_k), z(q_k))\| \to \infty$ as $q_k \to q$, $q \in \partial R^{2n}_{+} \setminus \{0\}$. (y, z) is bounded from below, however, for all $q \in R^{2n}_{++}$.

A.1 has been shown by Debreu (1972) and Mas-Colell (1974) to entail little loss of generality. A.2 implies that each consumer faces an ordinary budget constraint in the two periods of his life. As we later show, this is equivalent to assuming a fixed (possibly zero or negative) stock of fiat money. A.3 is standard. As we shall see, A.4 is used only to guarantee the existence of interior steady states. Although the theory can be extended to allow free goods, we do not attempt to do so here. Muller and Woodford (1983) have, in fact, extended the type of results presented in this paper to economies with general activity analysis production technologies that include free disposal and allow free goods.

The space of feasible economies $\mathscr{E}$ are the pairs (y, z) which satisfy A.1–A.4. This is a topological space in the weak C^1 topology described for example, by Hirsch (1976). Roughly, two economies (y^1, z^1) and (y^2, z^2) are close if the functions and their first derivatives are close.

A.1–A.4 are naturally satisfied by any demand function derived by aggregating the individual demand functions of utility maximizing consumers. Furthermore, Debreu (1974) has demonstrated that, for any (y, z) that satisfy A.1–A.3 and any compact subset of R^{2n}_{++}, there exists a generation of $2n$ utility maximizing consumers whose aggregate excess demands (y^*, z^*) agree with (y, z) on that subset. Since homogeneity allows us to restrict our attention to prices $q \in R^{2n}_{++}$ that satisfy such a price normalization as $\sum_{i=1}^{2n} q^i = 1$, this means that problems can occur only as some relative prices approach zero. As we point out in the next section, however, this minor technical problem plays no role in our study of steady states or of equilibrium price paths near steady states. Consequently, we are justified in viewing A.1–A.4 as completely characterizing demand functions derived from utility maximization by heterogenous consumers.

3. Steady states

A *steady state* of an economy $(y, z) \in \mathscr{E}$ is a relative price vector $p \in R^n_{++}$ and price level growth factor $\beta > 0$ such that

$$z(p, \beta p) + y(\beta p, \beta^2 p) = z(p, \beta p) + y(p, \beta p) = 0. \tag{3.1}$$

In other words, if relative prices in each period are given by p and the price level grows at β, the market is always in equilibrium. Since, if claims to good i now cost p^i then claims to good i next period cost βp^i, $1/\beta - 1$ is the steady state rate of interest.

Notice that any steady state price vector $(p, \beta p)$ is a special case of a price vector $q \in R^{2n}_{++}$ that satisies $z(q) + y(q) = 0$. We are now in a position to argue that, for our purposes, A.1–A.4 completely characterize excess demand functions derived from utility maximization, and that we need not worry about problems near the boundary of R^{2n}_{+}: if (y, z) satisfies A.1–A.3 and $\varepsilon > 0$, then there exists (y^*, z^*), derived from utility maximization by $2n$ consumers, that agrees with (y, z) on $S_\varepsilon = \{q \in R^{2n} \mid q'e = 1, q^i \geqq \varepsilon\}$. Here $e = (1, \ldots, 1)$. If (y, z) satisfies A.4, then as $q_k \to q$, $q \in \partial R^{2n}_{+} \setminus \{0\}$, $e'(z(q) + y(q)) \to \infty$. Consequently, S_ε can be chosen large enough so that $e'(z(q) + y(q)) > 0$ for all $q \in S_0 \setminus S_\varepsilon$. This obviously implies that we can choose S_ε large enough so that every steady state of (y, z) lies in its interior. Mas-Colell (1977) has further demonstrated that, for any $\varepsilon > 0$ and any (y, z) that satisfies A.1–A.4 and the condition that $e'(z(q) + y(q)) > 0$ for all $q \in S_0 \setminus S_\varepsilon$, there exists (y^*, z^*), derived from utility maximization by $2n$ consumers, that agrees with (y, z) on S_ε and also satisfies $e'(z^*(q) + y^*(q)) > 0$ for all $q \in S_0 \setminus S_\varepsilon$. Consequently, the only steady states of (y^*, z^*) are those of (y, z). Furthermore, S_ε can be chosen large enough so that (y, z) and (y^*, z^*) agree on any open neighborhood of these steady states in S_0.

The nominal steady state savings for the entire economy are $\mu = -p'y(p, \beta p)$. There are two kinds of steady states: *real* steady states in which $\mu = 0$ and monetary, or *nominal*, steady states in which $\mu \neq 0$. Gale (1973) refers to real steady states as *balanced*. By Walras's law, $p'(y+\beta z)=0$, which implies $\beta p'z = -p'y = \mu$. By the equilibrium condition, $p'(z+y)=0$, which implies $p'z=\mu$. Consequently, $(\beta - 1)\mu = 0$, and in a monetary steady state the interest rate must be zero. We shall see that a real steady state has $\beta = 1$ purely by coincidence. We therefore refer to a steady state with $\beta = 1$ as a nominal steady state. Gale refers to these as *golden rule* steady states since they maximize a weighted sum of utilities subject to the steady state consumption constraint.

We now examine the number of steady states. We first separate the nominal and real cases. We show that generically $\beta = 1$ and $\mu = 0$ do not both occur at the same steady state. If both $\beta = 1$ and $\mu = -p'y = 0$ at a steady state, then

$$\begin{aligned} &z(p,p) + y(p,p) = 0, \\ &-p'y(p,p) = 0. \end{aligned} \tag{3.2}$$

By virtue of Walras's law, the first n equations may be viewed as a system of $n-1$ equations while, by homogeneity, p constitutes $n-1$ independent variables. (3.2) may therefore be regarded as n equations in $n-1$ unknowns. Let us assume that

R.1. System (3.2) has no solution.

The importance of this regularity assumption is that it is satisfied by almost all $(y, z) \in \mathscr{E}$. Here 'almost all' means an open dense subset of $\mathscr{E}$. We call a property *generic* if it is satisfied by an open dense subset of a topological space. Note that we can easily show that genericity in $\mathscr{E}$ is equivalent to genericity in the space of excess demand functions derived from utility maximization (see the discussion of the boundary condition above). This has implications for economies parameterized by utility functions and endowments [see Mas-Colell (1974)]. The principal tool that we use to prove genericity is the following result from differential topology [see Guillemin and Pollack (1974, pp. 67–69)].

Transversality Theorem. Let M, V, N be smooth manifolds where $\dim M = m$ *and* $\dim N = n$. *Let $y \in N$. Suppose that $f: M \times V \to N$ is a C^r map, where $r > \max[0, m-n]$, such that for every (x, v) that satisfies $f(x,v) = y$, rank* $Df(x, v) = n$; *then the set of $v \in V$ for which $f(x,v) = y$ implies rank* $D_1 f(x,v) = n$ *has full Lebesgue measure. In other words, if y is a regular value of f, then, for all $v \in V$ in a set of full Lebesgue measure, it is a regular value of f_v.*

Since a set of full Lebesgue measure is dense, we can use this theorem to prove the density of sets that satisfy some property. Openness usually follows trivially from definitions. Notice that, since $Df=[D_1f\ D_2f]$, it suffices to demonstrate that rank $D_2f(x,v)=n$ to prove that, for almost all $v\in V$, $Df_v(x)$ has rank n whenever $f_v(x)=y$.

Proposition 3.1. The set of economies that satisfy R.1 is open and dense in $\mathscr{E}$.

Proof. Openness is obvious. To prove density, we let $v_1\in R^n$, $v_2\in R$ and construct the perturbation

$$y_v^i=y^i+\frac{\sum_{j=1}^{n}p_1^jv_1^j}{\sum_{j=1}^{n}p_1^j}-v_1^i+\frac{p_2^i}{p_1^i}v_2,\qquad (3.3)$$

$$z_v^i=z^i-v_2.$$

A check shows that, for v small enough, $(y_v,z_v)\in\mathscr{E}$; in other words, A.1–A.5 are satisfied. To show the set of economies that satisfy R.1 is dense, it suffices by the transversality theorem to show that the derivative of the system in (3.2) with respect to v has rank n at any solution: the only way 0 can be a regular value of $f_v(p)=(z_v(p,p)+y_v(p,p),\ -p'y_v(p,p))$ is for there to be no p for which $f_v(p)=0$. This derivative is

$$\begin{bmatrix} ep'-I & 0\\ 0 & -1\end{bmatrix}$$

for any $p\in S_\varepsilon$ where S_ε is now the set $\{p\in R^n\,|\,p'e=1,p^i\geqq\varepsilon\}$. This matrix has rank n as required. Q.E.D.

Nominal steady states are characterized by $z(p,p)+y(p,p)=0$. Since $z(p,p)+y(p,p)$ has the formal properties of the excess demand function of a static exchange economy with n goods, the theory of nominal steady states carries over directly from the static theory. For the sake of completeness we prove the following proposition:

Proposition 3.2. Every economy $(y,z)\in\mathscr{E}$ has a steady state in which $\beta=1$.

Proof. Choose ε small enough so that $e'(z(p,p)+y(p,p))>\alpha$ for all $p\in S_0\backslash S_\varepsilon$ and fixed $\alpha>0$. S_ε is obviously compact, convex, and, choosing $\varepsilon<1/n$, non-empty. For any $p\in S_\varepsilon$, define $f(p)$ as the vector in S_ε that is closest to $p+z(p,p)+y(p,p)$ in terms of euclidean distance. $f\colon S_\varepsilon\to S_\varepsilon$ is obviously continuous and, hence, by Brouwer's fixed point theorem, has a fixed point.

At any $p \in S_\varepsilon$ $f(p)$ is the vector that solves the problem of minimizing $1/2\|f-p-z(p,p)-y(p,p)\|^2$ subject to the constraints $f \geqq \varepsilon e$ and $f'e=1$. By the Kuhn–Tucker theorem such a point satisfies

$$\begin{aligned} &f-p-z(p,p)-y(p,p)-\lambda_1+\lambda_2 e=0, \\ &(f-\varepsilon e)'\lambda_1=0 \end{aligned} \tag{3.4}$$

for some $\lambda_1 \in R^n_+$, $\lambda_2 \in R$. At a fixed point $f=p$. Pre-multiplying the top line of (3.4) by $(p-\varepsilon e)'$ yields $(1-n\varepsilon)\lambda_2=-e'(z(p,p)+y(p,p))$; premultiplying by p' yields $\lambda_2=p'\lambda_1$. If $p \in \partial S_\varepsilon$, then $0>-\varepsilon\alpha \geqq (1-n\varepsilon)\lambda_2=(1-n\varepsilon)p'\lambda_1 \geqq 0$, which is impossible. Consequently, since any fixed point p lies in the interior of S_ε, $\lambda_1=0$, which implies $\lambda_2=0$, and (3.4) is the steady state condition. Q.E.D.

Following Debreu (1970), we impose the regularity assumption

R.2. $D_1z(p,p)+D_2z(p,p)+D_1y(p,p)+D_2y(p,p)$ *has rank* $n-1$ *at nominal steady states.*

To show that this condition is generic (open and dense) we apply a device that we use repeatedly in this paper. If $\mathscr{E}^1$ and $\mathscr{E}^2$ are topological spaces and $f:\mathscr{E}^1 \to \mathscr{E}^2$ is a continuous (inverse images of open sets are open) open (images of open sets are open) mapping it follows from unraveling definitions that, if $\hat{\mathscr{E}}^2 \subset \mathscr{E}^2$, then $f^{-1}(\hat{\mathscr{E}}^2)$ is beneric in $\mathscr{E}^1$ if and only if $\hat{\mathscr{E}}^2$ is generic in $\mathscr{E}^2$. Continuity of f allows us to conclude $f^{-1}(\hat{\mathscr{E}}^2)$ is open. The assumption that f is an open mapping means that a close approximation to a point in $\mathscr{E}^2$ is the image of a close approximation to the inverse image of that point in $\mathscr{E}^1$. This enables us to conclude that $f^{-1}(\hat{\mathscr{E}}^2)$ is dense. Applying this principle, since the map from $\mathscr{E}$ to static exchange economies with n goods is obviously a continuous open map, and R.2 is unknown to be generic for static exchange economies, we see that R.2 is generic in $\mathscr{E}$.

We can use the fixed point index theorem developed by Dierker (1972) to prove that R.2 implies that there is an odd number of nominal steady states. Let $J=D_1z+D_2z+D_1y+D_2y$, evaluated at a nominal steady state p. If we define index$(p)=\operatorname{sgn}(\det[-\bar{J}])$, where $\bar{J}$ is the $(n-1)\times(n-1)$ matrix formed by deleting the first row and column from J, then index theory implies that $\sum \text{index}(p)=+1$, where the sum is over all nominal steady states. For example, if (y,z) exhibits gross substitutability, which implies that $\det[-\bar{J}]>0$, then there is a unique nominal steady state.

Real steady states are characterized by the equations

$$\begin{aligned} &z(p,\beta p)+y(p,\beta p)=0, \\ &-p'y(p,\beta p)=0. \end{aligned} \tag{3.5}$$

Walras's law implies that $p'z(p,\beta p)=0$ at the steady state and, consequently, that (p,β) solves (3.5) if and only if it solves

$$(I-ep')(z(p,\beta p)+y(p,\beta p))=0, \qquad (3.6)$$
$$-p'y(p,\beta p)=0.$$

Proposition 3.3. Every economy has a steady state in which $\mu=-p'y(p,\beta p)=0$.

Proof. The proof of this proposition is similar to that of Proposition 3.2. We find a non-empty, compact, convex set whose interior contains all steady states that satisfy (3.5). We then define a continuous mapping of this set into itself whose fixed points are steady states.

We begin by putting bounds on p: A.4 implies that there exists some $\varepsilon>0$ such that $e'(z(p,\beta p)+y(p,\beta p))>\alpha>0$ for all $p\in S_0\backslash S_\varepsilon$ and all $\beta>0$. To see why, suppose instead that there exists a sequence $(p_k,\beta_k)\in S_0\times R_{++}$ such that $e'(z(p_k,\beta_k p_k)+y(p_k,\beta_k p_k))\leqq\alpha$ and $p_k\to p\in\partial S_0$. Now there is either a subsequence of (p_k,β_k) for which β_k converges or one for which $1/\beta_k$ converges. In the first case, the associated subsequence $(p_k,\beta_k p_k)$ provides an example of a price sequence that converges to a point on the boundary of R^{2n}_+ and violates A.4. In the second case, $((1/\beta_k)p_k,p_k)$ provides such an example. Consequently, we can find an $\varepsilon>0$ such that all steady states (p,β) have $p\in S_\varepsilon$. It is now easy to put bounds on β: A.4 implies that, for any $p\in S_\varepsilon, p'z(p,\beta_k p)\to\infty$ as $\beta_k\to 0$ and, similarly, $p'y((1/\beta_k)p,p)\to\infty$ as $\beta_k\to\infty$. Since S_ε is compact, we can find some $\bar\beta>0$ such that $-p'y(p,\beta p)>0$ for all $\beta\geqq\bar\beta$ and all $p\in S_\varepsilon$ and some $0<\underline{\beta}<\bar\beta$ such that $-p'y(p,\beta p)<0$ for all $\beta\leqq\underline{\beta}$ and all $p\in S_\varepsilon$.

Consider now the set $S_\varepsilon\times[\underline{\beta},\bar\beta]$. It is non-empty, compact and convex. Furthermore steady states that satisfy (3.5), if any exist, lie in its interior. For any $(p,\beta)\in S_\varepsilon\times[\underline{\beta},\bar\beta]$ we define $f(p,\beta)$ as the vector in $S_\varepsilon\times[\underline{\beta},\bar\beta]$ that is closest to $[p+(I-ep')(z(p,\beta p)+y(p,\beta p)),\ \beta-p'y(p,\beta p)]$ in terms of euclidean distance. Again using the Kuhn–Tucker theorem to characterize $f(p,\beta)$, we establish that any fixed point $(p,\beta)=f(p,\beta)$ must satisfy

$$-(I-ep')(z(p,\beta p)+y(p,\beta p))-\lambda_1+\lambda_2 e=0,$$
$$p'y(p,\beta p)-\lambda_3+\lambda_4=0, \qquad (3.7)$$
$$(p-\varepsilon e)'\lambda_1=0,\quad (\beta-\underline{\beta})\lambda_3=0,\quad (\bar\beta-\beta)\lambda_4=0$$

for some $\lambda_1\in R^n_+$, $\lambda_2\in R$, and $\lambda_3,\lambda_4\in R_+$. The choice of $\underline{\beta}$ and $\bar\beta$ implies that $\lambda_3=\lambda_4=0$. An argument identical to that in Proposition 3.2 implies $\lambda_1=0$ and $\lambda_2=0$. Consequently, a fixed point of f, which necessarily exists, is a steady state in which $\mu=-p'y(p,\beta p)=0$. Q.E.D.

The relevant regularity condition is

R.3. $$\begin{bmatrix} (I-ep')(\mathrm{D}_1 z+\beta \mathrm{D}_2 z+\mathrm{D}_1 y+\beta \mathrm{D}_2 y) & (I-ep')(\mathrm{D}_2 z+\mathrm{D}_2 y)p \\ -y'-p'(\mathrm{D}_1 y+\beta \mathrm{D}_2 y) & -p'\mathrm{D}_2 yp \end{bmatrix} \text{ has rank } n.$$

Since $S_\varepsilon \times [\underline{\beta}, \bar{\beta}]$ is compact, a standard argument implies that economies that satisfy R.3 at every real steady state have only a finite number of real steady states. Define index(p, β) to be $+1$ or -1 according to whether the sign of the determinant of the negative of the above matrix with its first row and column deleted is positive or negative. Another standard argument then implies that $\sum \text{index}(p, \beta) = +1$ when summed over all equilibria. This implies there is an odd number of real steady states, and indeed a unique real steady state if index$(p, \beta) = +1$ at every possible steady state.

Proposition 3.4. Given R.1, R.3 is also generic.

Proof. The openness of R.3 is immediate from the stability of transversal intersections and the continuity of the derivatives of (y, z). To prove density, we use the same perturbation as that used in the proof of Proposition 3.1. Differentiating the system in (3.6) with respect to v, we obtain

$$\begin{bmatrix} ep'-I & (I-ep')(\beta-1)e \\ 0 & -\beta \end{bmatrix}$$

at a steady state (p, β). Since this matrix has rank n, the proposition now follows from the transversality theorem. Q.E.D.

Let $\mathscr{E}^{\mathrm{R}}$ be the subset of $\mathscr{E}$ that satisfies R.1–R.3. We can summarize the discussion with the following result.

Proposition 3.5. $\mathscr{E}^{\mathrm{R}}$ is open dense in $\mathscr{E}$. Every economy in $\mathscr{E}^{\mathrm{R}}$ has an odd number of real steady states and an odd number of nominal steady states. No real steady state has $\beta = 1$. Furthermore, the number of steady states of each type is constant on connected components of $\mathscr{E}^{\mathrm{R}}$, and the steady states themselves vary continuously with the economy.

Suppose we want to show that for a generic economy certain properties are satisfied at all steady states. Mathematically, it is more convenient to prove that for a generic economy these properties are satisfied at a particular steady state. A useful fact about regular economies is that the latter property implies the former. To formalize this let

$$\mathscr{F}^{\mathrm{R}} \subset \mathscr{E}^{\mathrm{R}} \times S_\varepsilon \times [\underline{\beta}, \bar{\beta}]$$

be the set of (y, z, p, β) for which (p, β) is a steady state of (y, z). Let $\mathscr{F}^{G}$ be open dense in $\mathscr{F}^{R}$. Define $\mathscr{E}^{G}$ to be the subset of $\mathscr{E}^{R}$ such that, if $(y, z) \in \mathscr{E}^{G}$ and $(y, z, p, \beta) \in \mathscr{F}^{R}$, then $(y, z, p, \beta) \in \mathscr{F}^{G}$. It follows directly from Proposition 3.5 and the fact that finite intersections of open dense sets are open dense that $\mathscr{E}^{G}$ is open dense in $\mathscr{E}^{R}$. Consequently, in the sequel, we prove all theorems about genericity in $\mathscr{F}^{R}$, with the understanding that this carries over into $\mathscr{E}$.

4. Restrictions on demand derivatives

After characterizing steady states the natural next step is to study the behavior of equilibrium price paths near steady states. This is done by linearizing the equilibrium conditions of the steady state and studying the resulting linear difference equation. The qualitative dynamics of this system depend on the demand derivatives $\mathrm{D}_1 y$, $\mathrm{D}_2 y$, $\mathrm{D}_1 z$ and $\mathrm{D}_2 z$ evaluated at the steady state (p, β). The most convenient way to study these derivatives is to introduce the jet mapping $d\colon \mathscr{F} \to \mathscr{D}$ where $\mathscr{D}$ is a subset of the space of six-tuples $(\mathrm{D}_1 y, \mathrm{D}_2 y, \mathrm{D}_1 z, \mathrm{D}_2 z, p, \beta)$ and the mapping d applied to (y, z, p, β) yields the excess demand derivatives evaluated at (p, β).

What restrictions should we place on the elements of $\mathscr{D}$? Differentiating Walras's law, we see that

$$
\begin{aligned}
&y' + p'\mathrm{D}_1 y + \beta p'\mathrm{D}_1 z = 0, \\
&z' + p'\mathrm{D}_2 y + \beta p'\mathrm{D}_2 z = 0.
\end{aligned}
\tag{4.1}
$$

But the steady state condition says that $z' + y' = 0$. Consequently, we can rewrite Walras's law as

$$
p'(\mathrm{D}_1 y + \mathrm{D}_2 y + \beta \mathrm{D}_1 z + \beta \mathrm{D}_2 z) = 0. \tag{4.2}
$$

Differentiating the homogeneity assumption, we can rewrite it as

$$
\begin{aligned}
&(\mathrm{D}_1 y + \beta \mathrm{D}_2 y)p = 0, \\
&(\mathrm{D}_1 z + \beta \mathrm{D}_2 z)p = 0.
\end{aligned}
\tag{4.3}
$$

Now let us restrict attention to economies with steady states in $S_\varepsilon \times [\underline{\beta}, \bar{\beta}]$. We define $\mathscr{D}$ to be the six-tuples that satisfy (4.2) and (4.3) and for which $(p, \beta) \in S_\varepsilon \times [\underline{\beta}, \bar{\beta}]$. The following theorem implies that the space $\mathscr{D}$ captures all the important restrictions on demand derivatives.

Proposition 4.1 *The jet mapping d is a continuous open mapping of $\mathscr{F}^{R}$ onto an open dense subset of $\mathscr{D}$.*

Discussion. The idea is to make $d \in \mathscr{D}$ into a linear demand function near steady state and average this with the original demand to obtain a sequence $(y^k, z^k) \to (y, z)$ where $d(y, z, p, \beta) = d$. Since linear functions do not satisfy homogeneity and Walras's law, (y^k, z^k) is defined first to have $2n-1$ components that are linear functions of $2n-1$ prices. Homogeneity and Walras's law extend this to $2n$ components in $2n$ prices. Then (4.2) and (4.3) are used to show that the construction works, that is, $d(y^k, z^k, p^k, \beta^k) = d^k \to d$. The same general idea gives a global as well as local inverse.

Proof. Continuity of d is obvious. To prove the remainder of the proposition we need to know how to convert elements of $\mathscr{D}$ into elements of $\mathscr{F}^{\mathrm{R}}$. Suppose $d \in \mathscr{D}$. Let us normalize prices $q \in R^{2n}_{++}$ by setting $q^1 = 1$. Let $\bar{X}_d$ be the matrix of demand derivatives with first row and column deleted. Using (4.1), we see that we should define $y' = -p'(\mathrm{D}_1 y + \beta \mathrm{D}_1 z)$ and $z' = -p'(\mathrm{D}_2 y + \beta \mathrm{D}_2 z)$. Let $\bar{q}$ be the vector $(p, \beta p)$ with the first component deleted, and let $\bar{x}_d(\bar{q})$ be the vector (y, z) with the first component deleted. Let $\bar{q}_t$ be an arbitrary $2n-1$ vector. We define the linear affine function $\bar{x}_d : R^{2n-1} \to R^{2n-1}$ by the rule $\bar{x}_d(\bar{q}_t) = \bar{x}_d(\bar{q}) + \bar{X}_d(\bar{q}_t - \bar{q})$. Suppose that $x \in \mathscr{E}$ and that $\bar{x}$ is the last $n-1$ components of x viewed as a function on R^{2n-1}_{++} by setting $q^1 = 1$. We define $\bar{x}_\lambda$ to be the weighted average

$$\bar{x}_\lambda(\bar{q}_t) = \lambda(\bar{q}_t)\bar{x}_d(\bar{q}_t) + (1 - \lambda(\bar{q}_t))\bar{x}(\bar{q}_t). \tag{4.4}$$

Let $B \subset R^{2n}_{++}$ be the open ball of radius $\varepsilon > 0$ around q. We can construct $\lambda : R^{2n-1} \to R$ so that it is C^1 and satisfies $0 \leqq \lambda(\bar{q}_t) \leqq 1$, $\lambda(\bar{q}) = 1$, and $\lambda(\bar{q}_t) = 0$ for $\bar{q}_t \notin B$. Furthermore, we can choose λ so that $\mathrm{D}\lambda(\bar{q}) = 0$ and $\|\mathrm{D}\lambda(\bar{q}_t)\| < 3/\varepsilon$ [see Hirsch (1976, pp. 41–42)]. Consequently, $\bar{x}_\lambda$ coincides with $\bar{x}$ outside of B, but $\bar{x}_\lambda(\bar{q}) = \bar{x}_d(\bar{q})$ and $\mathrm{D}\bar{x}_\lambda(\bar{q}) = \bar{X}_d$. There is a unique extension of $\bar{x}_\lambda$ to $x_\lambda : R^{2n}_{++} \to R^{2n}$ that satisfies Walras's law and homogeneity. Furthermore, for ε small enough, the boundary assumption is satisfied. Consequently, we may assume $x_\lambda \in \mathscr{E}$. Finally, a direct computation shows that $d(x_\lambda, p, \beta) = d$. This shows how to convert element $d \in \mathscr{D}$ into elements $x_\lambda \in \mathscr{E}$.

Let us first use this construction to show that d is open. Let $d = d(x, p, \beta)$, let $d^k \to d$, and let $\varepsilon^k = \max\{\|q^k - q\|, \|\bar{x}^k_d(\bar{q}^k) - \bar{x}(\bar{q})\|, \|\bar{X}^k_d - \mathrm{D}\bar{x}(\bar{q})\|\}$. Then $\varepsilon^k \to 0$. Furthermore, a computation using the mean value theorem shows $x^k_\lambda \to x$. Since $\mathscr{E}^{\mathrm{R}}$ is open in $\mathscr{E}$, x^k_λ is eventually in $\mathscr{E}^{\mathrm{R}}$. This implies that d is open.

Next we show $d(\mathscr{F}^{\mathrm{R}})$ is dense in $\mathscr{D}$. Indeed, suppose $d \notin d(\mathscr{F}^{\mathrm{R}})$. Since $x_\lambda \in \mathscr{E}$, there is $x^k \to x_\lambda$ with $x^k \in \mathscr{E}^{\mathrm{R}}$. By construction, however, the steady state (p, β) is itself a regular state of x_λ in the ball B of fixed radius ε. Thus, x^k must have a steady state $(p^k, \beta^k) \to (p, \beta)$. Therefore, $(x^k, p^k, \beta^k) \in \mathscr{F}^{\mathrm{R}}$ and $d(x^k, p^k, \beta^k) \to d = d(x_\lambda, p, \beta)$. Q.E.D.

This result says that any generic set in $\mathscr{D}$ corresponds to a generic property in $\mathscr{E}$. Furthermore, any open set in $\mathscr{D}$ corresponds to a non-void open set in $\mathscr{E}$. It enables us to restrict our study entirely to the space $\mathscr{D}$.

It is of interest to see what R.1 and R.3 mean in $\mathscr{D}$. (4.1) implies that $p'y=0$ if and only if $p'(\mathrm{D}_1 y+\beta \mathrm{D}_1 z)p=0$. R.1 is therefore equivalent to the assumption that $p'(\mathrm{D}_1 y+\mathrm{D}_1 z)p=0$ implies $\beta \neq 1$. Let us define $J=\mathrm{D}_1 z+\beta \mathrm{D}_2 z+\mathrm{D}_1 y+\beta \mathrm{D}_2 y$. Homogeneity implies that $Jp=0$. At steady states where $\beta=1$ R.2 is equivalent to the assumption that J has rank $n-1$. At steady states where $\beta \neq 1$ Walras's law implies the matrix in R.3 equals

$$\begin{bmatrix} (I-ep')J & (I-ep')(\mathrm{D}_2 z+\mathrm{D}_2 y)p \\ \beta p'(\mathrm{D}_1 y-\mathrm{D}_2 z) & -p'\mathrm{D}_2 yp \end{bmatrix}.$$

A second application of Walras's law shows that this has the same rank as

$$\begin{bmatrix} J & (\mathrm{D}_2 z+\mathrm{D}_2 y)p \\ \beta p'(\mathrm{D}_1 y-\mathrm{D}_2 z) & -p'\mathrm{D}_2 yp \end{bmatrix}.$$

It also implies that if $Jx=0$ then $p'(\mathrm{D}_1 y-\mathrm{D}_2 z)x=0$. Consequently, R.3 implies that J has rank $n-1$. Observe that, if there is a vector x such that $x'J=0$ and $x'(\mathrm{D}_2 z+\mathrm{D}_2 y)p \neq 0$ and J has rank $n-1$, then R.3 is satisfied. It is straightforward to show that the former condition is generic given the latter.

5. Paths near steady states

A (perfect foresight) *equilibrium price path* is a finite or infinite sequence of prices $\{\ldots, p_{t-1}, p_t, p_{t+1}, \ldots\}$ such that $p_t \in R^n_{++}$ and

$$z(p_{t-1}, p_t)+y(p_t, p_{t+1})=0. \tag{5.1}$$

Our goal is to find generic conditions under which paths near steady states are well behaved, which means that they should follow a nice second order difference equation.

Fix a steady state (p, β). The equilibrium condition (5.1) can be linearized as

$$\begin{aligned} &\mathrm{D}_1 z(p_{t-1}-\beta^{t-1}p)+(\mathrm{D}_2 z+\beta^{-1}\mathrm{D}_1 y)(p_t-\beta^t p) \\ &\quad +\beta^{-1}\mathrm{D}_2 y(p_{t+1}-\beta^{t+1}p)=0. \end{aligned} \tag{5.2}$$

Here all derivatives are evaluated at $(p, \beta p)$ and we use the fact that excess demand derivatives are homogeneous of degree minus one. Suppose that the following condition holds.

R.4. D_2y is non-singular.

Then the linearized system can be solved to find

$$(q_{t+1}-\beta^{t+1}q)=G(q_t-\beta^t q), \tag{5.3}$$

where

$$G=\begin{bmatrix} 0 & I \\ G_1 & G_2 \end{bmatrix},$$

$G_1=-\beta D_2y^{-1}D_1z$, $G_2=-D_2y^{-1}(\beta D_2z+D_1y)$, $q=(p,\beta p)$ and $q_t=(p_{t-1},p_t)$. A direct implication of the implicit function theorem is

Proposition 5.1. If R.4 holds, then there is an open cone $U\subset R^{2n}_{++}$ around q and a unique function $g:U\to R^{2n}_{++}$, that is smooth, homogeneous of degree one, and such that

(*a*) *If $\{p_t\}$ is an equilibrium price path and $q_t, q_{t+1}\in U$, then $q_{t+1}=g(q_t)$.*
(*b*) *If $\{p_t\}$ has $q_t\in U$ at all times and $q_{t+1}=g(q_t)$, then it is an equilibrium price path. Furthermore,* $Dg(q)=G$.

Our goal is to establish that there are generic restrictions on the demand derivatives D_1y, D_2y, D_1z, D_2z such that R.4 holds and such that G is a nice matrix, and to prove that under these conditions g is a nice dynamical system. Since the qualitative properties of (5.3) are determined by the eigenvalues of G it is natural to ask what we can say about these. Homogeneity A.3 and (4.3) imply $Gq=\beta q$ and thus that one eigenvalue is β. Walras's law A.2 and (4.2) imply $p'[D_2yG_1D_2y]G=p'[D_2yG_1D_2y]$ and, since G and G' have the same eigenvalues, one eigenvalue is equal to one. The upshot is that since (4.2) and (4.3) are the only restriction on $\mathscr{D}$ these are the only restrictions on the eigenvalues. We prove this next.

6. Restrictions on the linearized system

We are interested in discovering the properties of the linearized system as represented by the matrix G. Consequently, we must translate the space of demand derivatives $\mathscr{D}$ into the space of dynamic matrices $\mathscr{G}$. It is convenient to work in the subset $\mathscr{D}^{\mathrm{R}}$ of $\mathscr{D}$ for which R.1–R.4 and the following restriction hold:

R.5. $K=D_1y+D_2y+\beta D_1z+\beta D_2z$ has rank $n-1$.

Note that Walras's law implies that $p'K=0$, so K cannot have full rank.

Proposition 6.1. $\mathscr{D}^{R}$ *is open dense in* $\mathscr{D}$.

Proof. Openness is obvious. To demonstrate the density of R.4, let us define $D_1y_v = D_1y + v\beta I$, $D_2y_v = D_2y - vI$, $D_1z_v = D_1z - v\beta I$, and $D_2z_v = D_2z + vI$. Leave p and β fixed. It is easy to verify that $(D_1y_v, D_2y_v, D_1z_v, D_2z_v, p, \beta)$ is an element of $\mathscr{D}$ if $(D_1y, D_2y, D_1z, D_2z, p, \beta)$ is. Let λ have the smallest absolute value of any non-zero, real eigenvalue of D_2y. Obviously, D_2y_v is non-singular for any v such that $0<|v|<\lambda$.

To demonstrate the density of R.5, let us define $D_1y_v = D_1y - v(I - ep')$ where $p'e = 1$. Observe that $(D_1y_v, D_2y, D_1z, D_2z, p, \beta)$ still satisfies the relevant versions of Walras's law, (4.2), and the homogeneity assumption, (4.3). Now $K_v = K + v(I - ep')$. Let $p_0' K_v = 0$. We know that $p'K = 0$. If p_0 is necessarily proportional to p, then K_v has rank $n-1$. But $(p_0' - (p_0'e)p')(K - vI) = 0$. Since $K - vI$ is non-singular for $v \neq 0$ small enough, $p_0 = (p_0'e)p$. Q.E.D.

Our next step, largely of technical importance, is to consider the mapping $h: \mathscr{D}^{R} \to \mathscr{H}$ where the elements of $\mathscr{H}$ are six-tuples $(D_1y, D_2y, G_1, G_2, p, \beta)$ that satisfy the appropriate conditions. The map h is the identity on the first two and last two components. G_1 and G_2 are defined as $G_1 = -\beta D_2y^{-1} D_1z$ and $G_2 = -D_2y^{-1}(\beta D_2z + D_1y)$. Since D_2y is non-singular on $\mathscr{D}^{R}$, h is obviously continuous. Equally important, it has a continuous inverse on $h(\mathscr{D}^{R})$ given by the identity on the first two and last two components and by

$$[D_1z\ D_2z] = -(1/\beta)[D_1y\ D_2y]G, \tag{6.1}$$

where

$$G = \begin{bmatrix} 0 & I \\ G_1 & G_2 \end{bmatrix}$$

as in (5.3).

Thus, h is a homeomorphism onto $\mathscr{H}^{R} = h(\mathscr{D}^{R})$. It remains to identify $\mathscr{H}^{R}$. Walras's law A.2 holds if and only if

$$p' D_2y[I - G_1 - G_2] = 0. \tag{6.2}$$

Note that this implies $p'[D_2yG_1\ \ D_2y]G = p'[D_2yG_1\ \ D_2y]$ and, therefore, that G has an eigenvalue equal to one. The homogeneity condition A.3 holds if and only if

$$(D_1y + D_2y)p = 0, \tag{6.3}$$

$$Gq = \beta q, \tag{6.4}$$

where $q=(p, \beta p)$. Consequently, G has an eigenvalue equal to β. R.4 is unchanged while R.5 becomes

$$I-G_1-G_2 \quad \text{has rank} \quad n-1. \tag{6.5}$$

(6.2)–(6.5) and R.4 completely characterize $\mathscr{H}^{\mathrm{R}}$.

Finally, we focus in on G itself, considering $\gamma: \mathscr{H}^{\mathrm{R}} \to \mathscr{G}$ where the elements of $\mathscr{G}$ are three-tuples of the form (G, p, β) and γ is the projection map. γ is obviously continuous; we want to show that it is an open map onto $\gamma(\mathscr{H}^{\mathrm{R}})$.

We examine (6.3) first. Since $\mathrm{D}_1 y$ does not appear except in this condition, $(\mathrm{D}_1 y+\beta \mathrm{D}_2 y)p=0$ serves only to determine $\mathrm{D}_1 y$ once $\mathrm{D}_2 y$ is given. Obviously, $\mathrm{D}_1 y$ may be locally chosen as a continuous function of β, $\mathrm{D}_2 y$, and p. The second condition is $Gq=\beta q$. The third condition is 6.5 which implies that G has a unit root.

We claim that this is all: (6.4) and (6.5) uniquely characterize $\mathscr{G}$, and γ is open. Thus, we must show (6.2) holds. Let x be in the left null space of $I-G_1-G_2$. We think of x as lying in the manifold formed by identifying radially opposite points on the unit sphere. Since $I-G_1-G_2$ has rank $n-1$, x is a continuous function of G. We need to be able to locally map vectors x and p continuously into non-singular matrices $\mathrm{D}_2 y$ such that $p' \mathrm{D}_2 y=x$. This, however, is obviously possible. We summarize our arguments with the following proposition:

Proposition 6.2. Let $\mathscr{G}$ be the space of (G, p, β) such that G has one unit root (counting geometric multiplicity), $Gq=\beta q$, and $I-G_1-G_2$ has rank $n-1$. Then the mapping of $\mathscr{D}^{\mathrm{R}}$ taking excess demand derivatives to coefficient matrices of the linearized system is continuous open and onto $\mathscr{G}$.

In particular, G is a coefficient matrix of a linearized system of a steady state q if and only if G has one unit root and $Gq=\beta q$. Small perturbations in G require only small perturbations in the demand derivatives and vice versa.

7. Restrictions on eigenvalues

We now examine the implication of the restrictions on G for its eigenvalues. It is convenient to work in the subspace $\mathscr{G}^{\mathrm{R}}$ of $\mathscr{G}$ for which $\beta^2 I-G_1-\beta G_2=\beta \mathrm{D}_2 y^{-1}(\mathrm{D}_1 z+\beta \mathrm{D}_2 z+\mathrm{D}_1 y+\beta \mathrm{D}_2 y)$ has rank $n-1$. Since this condition is already generic in $\mathscr{D}$, it is generic in $\mathscr{G}$. Let $\mathscr{S}$ be the manifold of eigenvalues of $2n \times 2n$ matrices: this is the subset of $2n$-tuples of complex numbers in which complex numbers occur only in conjugate pairs and in which vectors that differ only by the order of components are identified. The eigenvalue evaluation map σ maps $2n \times 2n$ matrices to $\mathscr{S}$ and is known to be continuous. We now consider the set $\mathscr{T} \subset \mathscr{S} \times [\underline{\beta}, \bar{\beta}]$ whose elements

(s, β) have exactly one component equal to one and one additional component equal to β *if* $\beta \neq 1$. We extend σ to $\tau: \mathscr{G}^{\mathrm{R}} \to \mathscr{T}$. We claim that the only restrictions on the eigenvalues of G are that one equal unity and one equal β. (If $\beta = 1$, these are only one restriction.) To justify this claim we use the following result:

Proposition 7.1. τ *is a continuous open mapping of* $\mathscr{G}^{\mathrm{R}}$ *onto an open dense subset of* $\mathscr{T}$.

Discussion. The idea is that, given G and its eigenvalues s, we can find $\tilde{G}$ corresponding to a small perturbation in s. The trick is to show that $\tilde{G}$ has the correct block form — that is, I in the upper right, 0 in the upper left. Since s is assumed to satisfy the restrictions of having eigenvalues β and 1, G_1 and G_2 automatically satisfy $Gq = \beta q$ and $I - G_1 - G_2$ singular.

To construct the local inverse we observe that a matrix of the correct form must have eigenvectors of the form $(h, s_i h)$. Conversely we show that, if the eigenvectors have the form $(h, \tilde{s}_i h)$, $\tilde{G}$ has the correct block form. Thus when s is perturbed we hold h fixed by perturbing the second component. These new eigenvectors and eigenvalues now yield a unique matrix $\tilde{G}$ that has the correct form.

Proof. τ is obviously continuous. To show τ is open, let $(s, \beta) = \tau(G, p, \beta)$, and suppose $(s^k, \beta^k) \to (s, \beta)$. We construct $G^k \to G$ with $\tau(G^k, p^k, \beta^k) = (s^k, \beta^k)$. Set $G^k = H^k C^k (H^k)^{-1}$. Given C^k, can we choose H^k so that G^k has the partitioned structure corresponding to a second order difference equation? Obviously G^k is the unique solution of $G^k H^k = H^k C^k$. Writing this out in partitioned form, we see that

$$\begin{bmatrix} 0 & I \\ G_1^k & G_2^k \end{bmatrix} \begin{bmatrix} H_{11}^k & H_{12}^k \\ H_{21}^k & H_{22}^k \end{bmatrix} = \begin{bmatrix} H_{21}^k & H_{22}^k \\ * & * \end{bmatrix}$$

$$= \begin{bmatrix} H_{11}^k & H_{12}^k \\ H_{21}^k & H_{22}^k \end{bmatrix} \begin{bmatrix} C_{11}^k & C_{12}^k \\ C_{21}^k & C_{22}^k \end{bmatrix} \tag{7.1}$$

$$= \begin{bmatrix} H_{11}^k C_{11}^k + H_{21}^k C_{12}^k & H_{11}^k C_{12}^k + H_{12}^k C_{22}^k \\ * & * \end{bmatrix},$$

from which it follows that G^k has the correct structure if and only if

$$\begin{aligned} H_{21}^k &= H_{21}(C^k, H_{11}^k, H_{12}^k) = H_{11}^k C_{11}^k + H_{12}^k C_{21}^k, \\ H_{22}^k &= H_{22}(C^k, H_{11}^k, H_{12}^k) = H_{11}^k C_{12}^k + H_{12}^k C_{22}^k. \end{aligned} \tag{7.2}$$

Now let H be a basis for R^{2n} such that $C=H^{-1}GH$ is in real canonical form. Obviously, $\sigma(C)=\sigma(G)=s$. Hirsch and Smale (1974, pp. 153–157) show how to construct a sequence of real matrices $C^k \to C$ with $\sigma(C^k)=s^k$. Set $H^k_{11}=H_{11}$, $H^k_{12}=H_{12}$ and H^k_{21}, H^k_{22} as defined above. By continuity $H^k \to H$ and is eventually non-singular, so G^k is well defined and, by construction, has the proper structure. Furthermore, since components of s^k are one and β, G^k has them as eigenvalues. Observe that, since G^k has a unit root, $I-G^k_1-G^k_2$ is singular, but, since $G^k \to G$, it has rank $n-1$. Next, the structure of G^k implies that there is an eigenvector corresponding to β^k that has the form $q^k=(p^k, \beta^k p^k)$. We think of this eigenvector as lying on the unit sphere with radial identification and thus being unique. Further, since $G^k \to G$, p^k is the unique component in the right null space of $\beta^2 I - G_1 - \beta G_2$ and, therefore, converges to p. Consequently, $(G^k, p^k, \beta^k) \to (G, p, \beta)$. Thus τ is an open mapping of $\mathscr{G}^{\mathrm{R}}$ into $\mathscr{T}$.

Finally, we want $\tau(\mathscr{G}^{\mathrm{R}})$ to be open dense in $\mathscr{T}$. Only density remains to be shown; we do this by constructing an open dense subset of $\mathscr{T}$, denoted $\mathscr{T}^{\mathrm{R}}$, such that $\mathscr{T}^{\mathrm{R}} \subset \tau(\mathscr{G}^{\mathrm{R}})$. Let $(s,\beta) \in \mathscr{T}$. We must give a generic condition on $\mathscr{T}$ that makes it possible to construct a matrix G for which (s,β) are the eigenvalues. Arranging diagonal blocks, we can construct a block diagonal matrix

$$C=\begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}, \tag{7.3}$$

in real canonical form where $\sigma(C)=s$ and where the first diagonal entry of C is β. We define $\mathscr{T}^{\mathrm{R}}$ to be the subset of $\mathscr{T}$ for which the above construction can yield a matrix C such that C_1-C_2 is non-singular. Clearly, $\mathscr{T}^{\mathrm{R}}$ is an open dense subset of $\mathscr{T}$. We need only show how to invert τ on $\mathscr{T}^{\mathrm{R}}$. Choose $p \in S_\varepsilon$, let H_{11} be a non-singular matrix with first column equal to p, and let $H_{12}=H_{11}$. Using (7.2), we set $H_{21}=H_{11}C_1$ and $H_{22}=H_{12}C_2$. Since C_1-C_2 is non-singular, so is

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{11}C_1 & H_{12}C_2 \end{bmatrix}. \tag{7.4}$$

Thus G is well-defined. Since C has only one unit eigenvalue, $I-G_1-G_2$ has rank $n-1$. Consequently, $(HCH^{-1}, p, \beta) \in \mathscr{G}$. Similarly, since C has only one eigenvalue equal to β $\beta^2 I - G_1 - \beta G_2$ has rank $n-1$, and in fact $(HCH^{-1}, p, \beta) \in \mathscr{G}^{\mathrm{R}}$. Q.E.D.

8. Nominal dynamics

Until now we have largely combined the study of real and nominal steady states. The dynamics near each type of steady state are, however, rather different. We begin by studying the nominal case. Here we know only that G has one unit root.

It is useful to define the money supply $m(q_t)=p_t'z(p_{t-1},p_t)$. This is homogeneous of degree one. Walras's law implies that it equals $-p_{t-1}'y(p_{t-1},p_t)$, and the equilibrium condition implies that $p_t'z(p_{t-1},p_t)=-p_t'y(p_t,p_{t+1})$. Consequently, $m(q_t)=m(g(q_t))$; the money supply is constant along equilibrium price paths. At a nominal steady state $\mu=m(q)\neq 0$. The homogeneity condition implies that, if $m(q_t)=\mu$, $\mathrm{D}m(q_t)q_t=\mu\neq 0$ and, therefore $m(q_t)=\mu$ defines a $2n-1$ submanifold $Q_\mu\subset R^{2n}_{++}$ that is transversal to the steady state ray and invariant under g. We denote the restriction of g to Q_μ by g_μ.

All interest focuses on g_μ. If $\operatorname{sgn}\mu_1=\operatorname{sgn}\mu_2$ then g_{μ_1} and g_{μ_2} exhibit the same dynamics except that the price level is increased by a factor of μ_1/μ_2. Examining the linearization, we see that $\mathrm{D}g_\mu$ is G restricted to $\mathrm{D}m(q)q_t=0$. Since Q_μ is invariant and transversal to the steady state ray, it follows that the generalized eigenspace of G that excludes the eigenvector q spans the space $\mathrm{D}m(q)q_t=0$ and that G restricted to this space has the eigenvalues of G excluding the one unit root known *a priori* to exist. Furthermore, the results of the previous section imply that the remaining eigenvalues are unrestricted. Let n^s be the number of these eigenvalues inside the unit circle. Using standard results, such as those in Irwin (1980), we can easily prove the following proposition:

Proposition 8.1. There is an open dense set of economies that satisfy the following conditions at all nominal steady states:

(a) *g_μ is a local diffeomorphism; that is, G is non-singular.*
(b) *g_μ has no roots on the unit circle; that is, g_μ is hyperbolic.*
(c) *g_μ has an n^s dimensional stable manifold W_s of $q_0\in Q_\mu$ for which $g_\mu^t(q_0)\to q$.*
(d) *g_μ has a $2n-n^s-1$ dimensional unstable manifold W_u of $q_0\in Q_\mu$ for which $g_\mu^{-t}(q_0)\to q$.*
(e) *(Hartmann's theorem) There is a smooth coordinate change $c(q)$ such that $c\circ g_\mu\circ c^{-1}=G$ on W_s, and for a residual set of economies this holds on all of Q_μ (and thus R^{2n}_{++}).*

One warning should be given about the genericity of these results: they hold for almost all economies when the only restrictions that we place on excess demands are A.1–A.4. Suppose, however, that we restrict our attention to economies with a single, two period lived consumer in each generation who has an intertemporally separable utility function. Then both D_2y and D_1z have at most rank one, and R.4 is violated. Since the set of economies that satisfy these restrictions is closed and nowhere dense, none of our previous analysis applies. Kehoe and Levine (1982b) analyze this case and show that it is essentially the same as that of an economy with one good in every period.

9. Real dynamics

We now study the neighborhood of a steady state $q=(p, \beta p)$ with $m(q)=0$ and $\beta \neq 1$. In this case prices are not stationary at a steady state, but grow or decline exponentially. Let $b: R^{2n}_{++} \to R$ be a function that is homogeneous of degree one. We can normalize prices to focus on the convergence of relative prices. Define g^b on $Q^b = \{q_t \in Q \mid b(q_t)=1\}$ by $g^b(q_t)=g(q_t)/b(g(q_t))$. If b is monotonically increasing, then it can be naturally thought of as a price index. As it is, it provides a one dimensional restriction on relative prices. Homogeneity implies that $b(g^b(q_t))=1$. We say that an equilibrium price path converges to q if $q_t/b(q_t) \to q$. This is true of a path beginning at q_0 if and only if the path under g^h starting at $q_0/b(q_0)$ converges to q.

What is the linear approximation to g^b? It is G^b $(1/\beta)(I - q'B)G$, where $B = \mathrm{D}b(q)$, restricted to $Bq_t = 0$. Choosing b so that $Bq_t=0$ defines the generalized eigenspace of G in which the eigenvector q is excluded, we see that the eigenvalues of G^b are those of $(1/\beta)G$, excluding the unit eigenvalue that arises from the eigenvalue β corresponding to q. One of these values is equal to $1/\beta$; the remaining $2n-2$ are unrestricted. Let $\bar{n}^s$ be the number of these remaining eigenvalues inside the unit circle. Then g^b generically is hyperbolic with an $\bar{n}^s$ dimensional stable manifold and a $2n-\bar{n}^s-1$ dimensional unstable manifold if $\beta<1$. Similarly g^b has a $\bar{n}^s+1$ dimensional stable manifold and a $2n-\bar{n}^s-2$ dimensional unstable manifold if $\beta>1$. Furthermore, g^b is linearizable by a smooth coordinate change on the stable manifold.

It is useful also to distinguish between initial conditions with $m(q_0)=0$ (real initial conditions) and those with $m(q_0) \neq 0$ (nominal initial conditions). Observe that $\mathrm{D}m(q)=(-p'\beta \mathrm{D}_1 z, p'\mathrm{D}_2 y)$, which, by R.4, generically does not vanish. Thus, generically $\mathrm{D}m(q_t)=0$ defines a $2n-1$ cone $Q_0 \subset R^{2n}_{++}$ invariant under g. This is transversal to Q^b and, consequently, intersects it in a $2n-2$ manifold Q^b_0 invariant under g^b. Furthermore, a simple computation shows that Q_0 is tangent to the eigenvectors of G except the one having the unit root; thus Q^b_0 is tangent to the eigenvectors of G^b except the eigenvector with root $1/\beta$. Since Q^b_0 is invariant and, for $q_t \in Q_0$, $m(q_t)=0$, nominal initial q_0 [those with $m(q_0) \neq 0$] can approach q only if $\beta>1$; otherwise, if $\beta<1$, nominal paths cannot approach the real steady state. On the other hand, in Q^b_0 the linearized system has all the eigenvalues of $(1/\beta)G$ except 1 and $1/\beta$. The real system on the invariant manifold Q^b_0 is, therefore, generically hyperbolic and has an $\bar{n}^s$ dimensional stable manifold and a $2n-\bar{n}^s-2$ dimensional unstable manifold. Furthermore, it is linearizable on the stable manifold.

10. Pareto efficiency and fiat money

Consider an infinite price sequence $\{p_1, p_2. p_3, \ldots\}$ that satisfies the conditions $(p_t, p_{t+1}) \in R^{2n}_{++}$ and

$$z_0(p_1) + y(p_1, p_2) = 0, \tag{10.1}$$

$$z(p_{t-1}, p_t) + y(p_t, p_{t+1}) = 0. \tag{10.2}$$

In other words, $\{p_1, p_2, p_3, \ldots\}$ is an equilibrium price path for the economy specified by the demand functions y and z and a demand function z_0 for the old generation alive in the first period. For such an economy, where each generation consists of a representative consumer, Balasko and Shell (1980) have established that a necessary and sufficient condition for Pareto efficiency is that the infinite sum $\sum 1/\|p_t\|$ diverges. They require that a certain uniform curvature condition on indifference surfaces be satisfied. This condition, while restrictive in non-stationary models, is naturally satisfied in a stationary model such as ours. This result can easily be extended to economies with many consumers in each generation. Consequently, steady states with a non-negative interest rate where $\beta \leqq 1$, are Pareto efficient. So are paths that converge to them. An economy always has a Pareto efficient steady state since it always has a steady state where $\beta = 1$. Is there anything more we can say? Can we, for example, guarantee the existence of a Pareto efficient steady state where $\mu \geqq 0$?

To answer these questions, let us rephrase the conditions that characterize a steady state. Consider pairs (p, β) that satisfy the price normalization $(p'e) = 1$. Let $f: S_\varepsilon \times [\underline{\beta}, \bar{\beta}] \rightarrow R^{n-1}$ be given by the first $n-1$ coordinate functions of $(I - ep')(z(p, \beta p) + y(p, \beta p))$. In other words,

$$f(p, \beta) = L(I - ep')(z(p, \beta p) + y(p, \beta p)), \tag{10.3}$$

where L is the projection operator that can be represented in standard coordinates by the $(n-1) \times n$ matrix.

$$L = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}. \tag{10.4}$$

We work with the function $(I - ep')(z + y)$ because, unlike $(z + y)$ itself, its first $(n-1)$ coordinates are equal to zero only if its last coordinate is equal to zero. This is because $p'(I - ep')(z(p, \beta p) + y(p, \beta p)) = 0$. Also, like $(z + y)$ itself, $(I - ep')(z + y)$ has the property that we can select $\varepsilon > 0$ small enough so that $e'(I - ep')(z(p, \beta p) + y(p, \beta p)) > \alpha > 0$ for all $p \in S_0 \backslash S_\varepsilon$ and any $\underline{\beta} \leqq \beta \leqq \bar{\beta}$. To see why, suppose instead that $e'(I - ep'_k)(z(p_k, \beta_k p_k) + y(p_k, \beta_k p_k)) \leqq 0$ for a sequence $(p_k, \beta_k) \rightarrow (p, \beta)$, $p \in S_0$. Since $e'(z(p_k, \beta_k p_k) + y(p_k, \beta_k p_k)) \rightarrow \infty$ and $z + y$

is bounded from below, this implies $p_k'(z(p_k, \beta_k p) + y(p_k, \beta_k p_k)) \to \infty$. This can only happen if $\beta \neq 1$. Walras's law can be used to rewrite this expression as either $(1-\beta_k)p_k' z(p_k, \beta_k p_k)$ or as $(1/\beta_k)(\beta_k - 1)p_k' y(p_k, \beta_k p_k)$. If $\beta > 1$, then $(1-\beta_k)p_k' z(p_k, \beta_k p_k)$ is bounded from below. Similarly, if $\beta < 1$, then $(1/\beta_k)(\beta_k - 1)p_k' y(p_k, \beta_k p_k)$ is bounded from above. In either case $p_k'(z(p_k, \beta_k p_k) + y(p_k, \beta_k p_k))$ is bounded from above, which is a contradiction.

In what follows, it is important that f be C^2. To ensure this, we assume that y and z are not only C^1 but also C^2. We need to assume that f is C^2 so that we can use the transversality theorem to prove that 0 is generically a regular value of f. Indeed, for $v \in R^{n+1}$, we define

$$f_v(p, \beta) = L(I - ep')(z_v(p, \beta p) + y_v(p, \beta p)), \tag{10.5}$$

where y_v and z_v are defined as in the proof of Proposition 3.1. Differentiating f_v with respect to v, we obtain the $n \times (n+1)$ matrix

$$\left[L\left(\frac{1}{e'p} ep' - I\right) \quad L(I - ep')(\beta - 1)e \right].$$

Notice that $x'[(1/e'p)ep' - I] = 0$ implies that x is a scalar multiple of p. Since $u'L = [u_1 \; u_2 \ldots u_{n-1} \; 0]$ for any $u \in R^{n-1}$, however, this implies that, for all $p \in S_\varepsilon$, $u'L[(1/e'p)ep' - I] = 0$ only if $u = 0$. Consequently, this matrix has rank $n-1$, and 0 is a regular value of f_v for all v in a subset of R^{n+1} of full Lesbesgue measure. It is now, as before, a straightforward matter to demonstrate that 0 is a regular value of f for all (y, z) in an open dense subset of $\mathscr{E}^{\mathrm{R}}$.

What does the pre-image of 0 under f look like? Obviously, $f^{-1}(0)$ is compact since $S_\varepsilon \times [\underline{\beta}, \bar{\beta}]$ is compact and f is continuous. Since $f(p, \beta)$ cannot equal zero for any p on the boundary of S_ε, the only points in $f^{-1}(0)$ on the boundary of $S_\varepsilon \times [\underline{\beta}, \bar{\beta}]$ are those where β equals $\underline{\beta}$ or $\bar{\beta}$. We have argued that 0 is generically a regular value of f on the interior of $S_\varepsilon \times [\underline{\beta}, \bar{\beta}]$. Our argument also implies that 0 is generically a regular value of f restricted to $S_\varepsilon \times \{\beta\}$ for almost all fixed β; in particular, 0 is generically a regular value of f on the boundary of $S_\varepsilon \times [\underline{\beta}, \bar{\beta}]$. Unfortunately, $S_\varepsilon \times [\underline{\beta}, \bar{\beta}]$ is not a smooth manifold with boundary because it has corners. Since $f^{-1}(0)$ stays away from these corners, however, it is a smooth one dimensional manifold with boundary whose boundary is contained in the boundary of $S_\varepsilon \times [\underline{\beta}, \bar{\beta}]$. Furthermore, using index theory we can show that $f(p, \beta) = 0$ has an odd number of solutions when $\beta = \underline{\beta}$ and an odd number of solutions when $\beta = \bar{\beta}$.

Define $m(p, \beta) = -p'y(p, \beta)$ for all $(p, \beta) \in f^{-1}(0)$. There are two distinct ways for $(p, \beta) \in f^{-1}(0)$ to be an equilibrium: $m(p, \beta) = 0$ or $\beta = 1$. In either case, Walras's law implies that $(z(p, \beta) + y(p, \beta))$ is equal to 0.

Consider now the graph of m, $\{(p, \beta, m) \in S_\varepsilon \times [\underline{\beta}, \bar{\beta}] \times R \,|\, f(p, \beta) = 0,\; m =$

$m(p, \beta)\}$. It is obviously a smooth one dimensional manifold with boundary diffeomorphic to $f^{-1}(0)$. Steady states of (y, z) are points where the graph of m intersects either the $n-1$ dimensional submanifold of $S_\varepsilon \times [\underline{\beta}, \bar{\beta}] \times R$ where $m=0$ or the $n-1$ dimensional submanifold where $\beta=1$. We can picture these intersections graphically if we project $S_\varepsilon \times [\underline{\beta}, \bar{\beta}] \times R$ onto $[\underline{\beta}, \bar{\beta}] \times R$. Under this projection the graph of m need not be embedded submanifold, of course, because it may contain points of self-intersection. It is, however, an immersed submanifold. The self-intersections are generically transversal, but this is not important for our arguments.

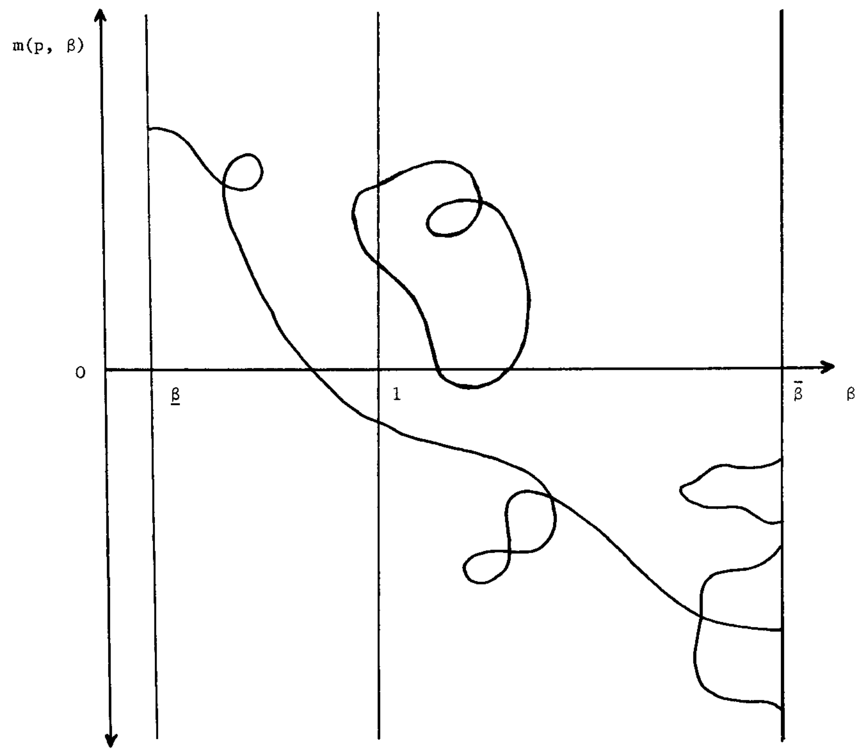

Fig. 1

R.1 says that the graph of m does not pass through $(1,0)$; R.2 says that it intersects the line $\beta=1$ transversally; and R.2 says that it intersects the line $m=0$ transversally. Considering diagrams like that in fig. 1, we can see why every economy does, in fact, have at least one steady state where $\beta \leqq 1$ and $\mu \geqq 0$. There is an odd number of points in $f^{-1}(0)$ where $\beta=\underline{\beta}$. Because of the boundary condition, $m(p, \beta)>0$ at all of these points. An even number, possibly zero, of these points are the endpoints of paths that return to the boundary $\beta=\underline{\beta}$. An odd number, at least one, must be endpoints of paths that lead to the boundary $\beta=\bar{\beta}$, where $m(p,\beta)<0$. Such a path must either cross the line $m=0$ where $\beta\leqq 1$ or cross the line $\beta=1$ where $m\geqq 0$. This same sort of argument can be used to demonstrate that every economy has at least one steady state where $\beta\geqq 1$ and $\mu\leqq 0$.

11. Many periods of life

Until now we have considered only the case in which consumers live two periods. Suppose instead they live m years $m>2$. The excess demand of cohort t in the jth year of life is denoted $x^j(p_t, p_{t+m-1})$. Assumptions A.1, A.3 and A.4 with the obvious notational changes are otherwise unchanged; Walras's law now has the form

$$A.2'.\quad \sum_{j=1}^{m} p'_{t+j-1}x^j(p_t,\ldots,p_{t+m-1})=0.$$

There are two approaches to the m-years of life case. One is to use a standard trick to reduce it to the two periods of life case. To do so we lump together $(m-1)$ years to form a period and $(m-1)$ cohorts to form a generation. Thus in period t there are now $n(m-1)$ commodities with prices

$$\hat{p}_\tau=(p_t,\ldots,p_{t+m-2}) \quad \text{where} \quad t=(m-1)(\tau-1)+1.$$

If previously there were l consumers in a cohort, there are now $l(m-1)$ consumers per generation. The excess demand of 'young' people in generation t for the kth block of n commodities is

$$y^k(\hat{p}_\tau,\hat{p}_{\tau+1})=\sum_{i=1}^{k} x^i(p_{t+k-1},\ldots,p_{t+m+k-i-1}), \tag{11.1}$$

and that of old people is

$$z^k(\hat{p}_\tau,\hat{p}_{\tau+1})=\sum_{i=1}^{m-k} x^{k+i}(p_{t+m-i-1},\ldots,p_{t+2m-i-2}), \tag{11.2}$$

where again $t=(m-1)(\tau-1)+1$. This restructuring of years and cohorts is shown in table 1.

This approach is useful, for it shows the essential unity of the cases $m=2$ and $m>2$. As in section 8 it enables us immediately to define

Table 1

Generation	Year Cohort	Period $t=1$: 1	$t=1$: $2 \ldots m-1$	$t=2$: m	$t=2$: $m+1 \ldots 2m-2$	$\ldots$
1	1	x^1	$x^2 \ldots x^{m-1}$	x^m	$0 \quad \ldots 0$	
	2	0	$x^1 \ldots x^{m-2}$	x^{m-1}	$x^m \quad \ldots 0$	
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots \quad \vdots$	
	$m-1$	0	$0 \ldots x^1$	x^2	$x^3 \quad \ldots x^m$	
2	m	0	$0 \ldots 0$	x^1	$x^2 \quad \ldots x^{m-1}$	
	$m+1$	0	$0 \ldots 0$	0	$x^1 \quad \ldots x^{m-2}$	
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots \quad \vdots$	
	$2m-2$	0	$0 \ldots 0$	0	$0 \quad \ldots x^1$	
	$\vdots$					

$$
\begin{aligned}
m(\hat{q}_\tau) &= \hat{p}'_\tau(\hat{p}_{\tau-1}, \hat{p}_\tau) \\
&= \sum_{k=1}^{m-1} p'_{t+k-1} \sum_{i=1}^{m-k} x^{k+i}(p_{t-i}, \ldots, p_{t+m-i-1}) \\
&= m(p_{t-m+1}, \ldots, p_{t+m-2}),
\end{aligned} \tag{11.3}
$$

and to conclude that this remains constant along paths. More generally, Walras's law and the equilibrium condition can be used to show that $m(p_{t-m+1}, \ldots, p_{t+m-2})$ is constant for all t, not merely for $t=(m-1)(\tau-1)$ as implied by the results of section 8.

There are two drawbacks of reducing the m year to the two period case. The first is that y and z defined above are nowhere dense in $\mathscr{E}$ since they satisfy many non-generic restrictions ($D_2 y$ is upper triangular for example). Thus the genericity results do not apply directly. Second, a steady state of (y, z) may actually be an m cycle of the m year economy; $\hat{p}_{t+1} = \beta \hat{p}_t$ implies only $p_{t+m-1} = \beta p_t$ and not $p_{t+1} = \beta^{1/(m-1)}$ as we require at a steady state.

The solution to these difficulties is to work directly with the equilibrium condition

$$
\sum_{j=1}^{m} x^j(p_{t-j+1}, \ldots, p_{t+m-j}) = 0, \tag{5.1'}
$$

viewing this as an $2(m-1)$ order implicit difference equation on R^n. The results of the previous sections then generalize with merely notational changes. We review the highlights.

As in section 3 a steady state satisfies $p_{t+1} = \beta p_t$. Thus, as in that section, (5.1′) results in $n+1$ equations in $n+1$ unknowns p and β. As before there are both real and nominal steady states as $\mu=m(p,\beta p,\beta^2 p,\ldots,\beta^{m-1}p)$ is zero or not. The existence proofs are virtually unchanged. The perturbation (3.3) used to establish that steady states are regular (a finite odd number of each type) and that $\mu=0$ and $\beta=1$ does not occur is now given by identifying x^1 with y with x^m and z. The required perturbation is just $x_v^1=y_v$ and $x_v^m=z_v$ the excess demands $x^2,\ldots,x^{m-1}$ are unchanged. This gives us the needed n degrees of freedom to establish that the steady state condition contains n independent equations.

Turning to behavior near a steady state we establish as in section 4 the restrictions on demand derivatives implied by Walras's law and homogeneity. Differentiating Walras's law, we find the analog of (4.1)

$$x^j+p'\sum_{i=1}^{m}\beta^{i-1}\mathrm{D}_j x^i=0 \tag{4.1'}$$

Using the equilibrium condition $\sum_{j=1}^{m}x^j=0$, we can derive restriction on demand derivatives as

$$p'\sum_{j=1}^{m}\sum_{i=1}^{m}\beta^{i-1}\mathrm{D}_j x^i=0. \tag{4.2'}$$

In addition, homogeneity implies that

$$\sum_{i=1}^{m}\beta^{i-1}\,\mathrm{D}_i x^j\,p=0. \tag{4.3'}$$

The proof of Proposition 4.1 that (4.2′) and (4.3′) are the only restrictions on demand derivatives now goes through with only notational changes.

Section 5, describing the relationship between the linear and non-linear system, is unchanged where the regularity condition is now

R.4′. $\mathrm{D}_m x^1$ is non-singular.

Of course $q_t=(p_{t-2m+3},\ldots,p_t)$, and the matrix G is

$$\begin{bmatrix} 0 & 0 & \ldots & 0 & I \\ 0 & 0 & \ldots & I & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & I & \ldots & 0 & 0 \\ G_1 & G_2 & \ldots & G_{2m-3} & G_{2m-2} \end{bmatrix},$$

where

$$G_i = -(\mathrm{D}_m x^1)^{-1}\left[\sum_{j=\max[m+1-i,\,1]}^{\min[2m-i,\,m]} \beta^{j-i}\,\mathrm{D}_{j+i-m}x^j\right]. \tag{11.4}$$

Arguing with notational modification as in section 6, we can establish that $\mathrm{D}_m x^1$ is generically non-singular and homogeneity (4.2′) and Walras's law (4.3′) imply only the G has a unit root and at a real steady state a root β.

Finally, the argument in section 7 that the remaining $2(m-1)n-1$ (nominal steady state) or $2(m-1)n-2$ (real steady state) roots are unrestricted can be extended. In section 7 the key was that G had the correct block structure if and only if the eigenvectors had the form $(h, s_i h)$ where s_i was the corresponding eigenvalue; in the m years of life case it is straightforward to show that G has the correct block structure if and only if the eigenvectors have the form $(h, s_i h, \ldots, s_i^{2m-3} h)$.

Having shown the linearized system generically has only one restriction on eigenvalues in the nominal case and two in the real case, we can establish the results of the remaining sections by collapsing the m year case to the two period case as described above.

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