Decision Theory: Time

Additive Separability

finite sum

$\sum_{t=1}^{T} u_t$

finite average

$(1/T)\sum_{t=1}^{T} u_t$

infinite time average

$$LIM(1/T)\sum_{t=1}^{T} u_t$$

where *LIM* could be liminf, limsup or Banach limit

with liminf and limsup there are two versions of expected utility:

ELIM vs *LIME* former makes sense, latter is actually used

with Banach limit it makes no difference

Impatience

discounted utility

$$\sum_{t=1}^{T} \delta^{t-1} u_t$$

infinite discounted utility

$$\sum_{t=1}^{\infty} \delta^{t-1} u_t$$

average discounted utility

$$(1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} u_t$$

note that average present value of 1 unit of utility per period is 1

The real equity premium puzzle

Utility $u(x) = \dfrac{x^{1-\rho}}{1-\rho}$, $\sum_{t=1}^{\infty} \delta^{t-1} u_t$

Consumption grows at a constant rate $x_t = \gamma^t$

$u'(x) = x^{-\rho}$

marginal rate of substitution $\dfrac{1}{1+r} = \dfrac{\delta u'(x_{t+1})}{u'(x_t)} = \dfrac{\delta \gamma^{-\rho(t+1)}}{\gamma^{-\rho t}} = \delta \gamma^{-\rho}$

1889-1984 from Shiller [1989]

average real US per capita consumption growth rate 1.8%

$\rho = 8.84 \;\; r = 17\%$

Mean real return on bonds 1.9%; Mean real return on S&P 7.5%

http://www.dklevine.com/econ201/interest.xls

How does the market react to good news?

Value of claims to future consumption relative to current consumption

$x_1 = 1$

$$\frac{\sum_{t=2}^{\infty} \delta^{t-1} u'(x_t) x_t}{u'(1)}$$

$$\sum_{t=2}^{\infty} \delta^{t-1} \gamma^{-(t-1)\rho} \gamma^{t-1} = \sum_{t=1}^{\infty} \left[\delta \gamma^{1-\rho} \right]^t = \frac{\delta \gamma^{1-\rho}}{1 - \delta \gamma^{1-\rho}}$$

to be finite we need $\delta \gamma^{-\rho} < 1$

$$\frac{\partial}{\partial \gamma} \frac{\delta \gamma^{1-\rho}}{1 - \delta \gamma^{1-\rho}} = \frac{\delta (1-\rho) \gamma^{-\rho}}{(1 - [\delta \gamma^{-\rho}])^2}$$

$\rho > 1$ this is negative

Hyperbolic Discounting

(based on Villaverde and Mukherji [2001])

Q1: would you like $10 today or $15 tomorrow?

Q2: would you like $10 100 days from now or $14 101 days from now?

Some people answer prefer $10 in Q1 and $14 in Q2. This is inconsistent with (geometric) discounting and a time and risk invariant marginal rate of substitution between days.

Note that (because of asset markets) this makes little sense when expressed in terms of money. So let us suppose that the "paradox" refers to consumption.

One explanation: "hyperbolic discounting" meaning preferences of the form $u(c_1) + \theta \sum_{t=2}^{\infty} \delta^{t-1} u(c_t)$

A more straightforward explanation:

Uncertainty about preferences 100 days from now.

Suppose marginal utility of consumption can take on two values 1 or 2 with equal probability and that the daily subjective discount factor is to a good approximation 1.

Today the value of todays and tomorrow's marginal utility is know with certainty. Hence the subjective interest rate can take on the values of 1, 0 or -½ with probabilities .25, .5 and .25. Expected subjective interest rate is .125 = 1/8. If you are offered 10 today versus 15 tomorrow, you take 10 today with probability .25.

Suppose on the other hand, suppose that preferences 100 days from now are unknown. Ratio of expected utilities is 1, so subjective interest rate is 0. If you are offered 10 in 100 days versus 14 in 101 days you always take 14.

Notice that pigeons have apparently figured this out correctly.

Experiments that have examined demand for commitment and consumption favor the geometric theory.

Dynamic Programming

$\alpha \in A$ action space: finite

$y \in Y$ state space: finite

$\pi(y' | y, \alpha)$ transition probability

period utility $u(\alpha, y)$ with discount factor $0 \leq \delta < 1$

$\overline{u} = \max u(\alpha, y) - \min(u(\alpha, y))$

Strategies

finite histories $h = (y_1, y_2, \ldots, y_t)$ with $t(h) = t$ $y(h) = y_t$; $h-1$; $y_1(h)$; $h' \geq h$

H space of all finite histories; this is countable

strategies $\sigma : H \rightarrow A$

Σ space of all strategies

all maps from a countable set to a finite set

the product topology $\sigma^n \rightarrow \sigma$ means that $\sigma^n(h) \rightarrow \sigma(h)$ for every h

Theorem: every sequence in the product topology has a convergent subsequence, so the space of strategies is compact

(proven in any elementary topology textbook) define a strong Markov strategy $\sigma(h) = \sigma(h')$ if $y(h) = y(h')$

a strong Markov strategy is equivalent to a map

$$\sigma : Y \to A$$

recursively define

$$\pi(h | y_1, \sigma) \equiv$$

$$\begin{cases} \pi(y(h) | y(h-1), \sigma(h-1))\pi(h-1 | y_1, \sigma) & t(h) > 1 \\ 1 & t(h) = 1 \text{ and } y_1(h) = y_1 \\ 0 & t(h) = 1 \text{ and } y_1(h) \neq y_1 \end{cases}$$

calculate the average present value of the objective function

$$V(y_1, \sigma) \equiv (1-\delta)\sum_{h \in H} \delta^{t(h)-1} u(\sigma(h), y(h))\pi(h | y_1, \sigma)$$

Dynamic Programming Problem

(*) maximize $V(y_1, \sigma)$ subject to $\sigma \in \Sigma$

a value function is a map $v: Y \rightarrow \Re$ bounded by $\bar{u}$

note that in this setting, it is simply a finite vector v_y

Lemma: a solution to (*) exists

Definition: ***the*** value function

$$v(y_1) \equiv \max_{\sigma \in \Sigma} V(y_1, \sigma)$$

Proof: the maximum exists because in the product topology on Σ $V(y_1, \sigma)$ is continuous in σ and Σ is compact why is V continuous?

suppose $\sigma^n \to \sigma$

$$V(y_1, \sigma^n) = (1-\delta)\sum_{h \in H} \delta^{t(h)-1} u(\sigma^n(h), y(h)) \pi(h | y_1, \sigma^n)$$

$$= (1-\delta)\sum_{t(h)<T} \delta^{t(h)-1} u(\sigma^n(h), y(h)) \pi(h | y_1, \sigma^n)$$

$$+(1-\delta)\sum_{t(h)\geq T} \delta^{t(h)-1} u(\sigma^n(h), y(h)) \pi(h | y_1, \sigma^n)$$

$$\to (1-\delta)\sum_{t(h)<T} \delta^{t(h)-1} u(\sigma(h), y(h)) \pi(h | y_1, \sigma) + O(\delta^T \overline{u})$$

so as $T \to \infty$ we have $V(y_1, \sigma^n) \to V(y_1, \sigma)$

Bellman equation

we define a map $T:\Re^{Y}\rightarrow\Re^{Y}$ by $w' = T(w)$ if

$$w'(y_1) = \max_{\alpha\in A}(1-\delta)u(\alpha,y_1) + \delta\sum_{y'_1\in S}\pi(y'_1|y_1,\alpha)w(y'_1)$$

Lemma: the value function is a fixed point of the Bellman equation $T(v) = v$

in other words the most you can get next period is also given by the value function

$$v(y_1) = \max_{\alpha\in A}(1-\delta)u(\alpha,y_1) + \delta\sum_{y'_1\in S}\pi(y'_1|y_1,\alpha)v(y'_1)$$

Lemma: the Bellman equation is a contraction mapping

$\|T(w) - T(w')\| \leq \delta \|w - w'\|$

Proof:

key observation $\|\max_\alpha f(\alpha) - \max_\alpha g(\alpha)\| \leq \max_\alpha \|f(\alpha) - g(\alpha)\|$

$$\left\| \max_{\alpha\in A}(1-\delta)u(\alpha,y_1) + \delta \sum_{y'_1\in S} \pi(y'_1 | y_1,\alpha) w(y'_1) - \max_{\alpha\in A}(1-\delta)u(\alpha,y_1) + \delta \sum_{y'_1\in S} \pi(y'_1 | y_1,\alpha) w'(y'_1) \right\|$$

$$\leq \max_{\alpha\in A} \left\| (1-\delta)u(\alpha,y_1) + \delta \sum_{y'_1\in S} \pi(y'_1 | y_1,\alpha) w(y'_1) - (1-\delta)u(\alpha,y_1) + \delta \sum_{y'_1\in S} \pi(y'_1 | y_1,\alpha) w'(y'_1) \right\|$$

$$= \max_{\alpha\in A} \left\| \delta \sum_{y'_1\in S} \pi(y'_1 | y_1,\alpha) w(y'_1) - \delta \sum_{y'_1\in S} \pi(y'_1 | y_1,\alpha) w'(y'_1) \right\|$$

$$\leq \delta \max_{\alpha\in A} \sum_{y'_1\in S} \pi(y'_1 | y_1,\alpha) \mid w(y'_1) - w'(y'_1) \mid$$

$$\leq \delta \max_{\alpha\in A} \sum_{y'_1\in S} \pi(y'_1 | y_1,\alpha) \|w - w'\|$$

$$= \delta \|w - w'\|$$

Corollary: the Bellman equation has a unique solution

Proof: Let w be another solution

$$\|v - w\| = \|T(v) - T(w)\| \leq \delta \|v - w\| \Rightarrow \|v - w\| = 0$$

Conclusion: the unique solution to the Bellman equation is the value function

since the value function is *a* solution to the Bellman equation, and the solution is unique

Lemma: there is a strong Markov optimum and it may be found from the Bellman equation

Proof:

define a strong Markov plan by

$$\sigma(y_1) \in \arg\max_{\alpha \in A}(1-\delta)u(\alpha, y_1) + \delta \sum_{y'_1 \in S(y_1)} \pi(y'_1 | y_1, \alpha) v(y'_1)$$

work the value function forward recursively to find

$$v(y_1(h)) = (1-\delta)\sum\nolimits_{t(h)<T} \delta^{t(h)-1} \pi(y(h)|y_1(h),\sigma) u(\sigma(y(h)), y(h))$$
$$+(1-\delta)\sum\nolimits_{t(h)=T} \delta^{T} \pi(y(h)|y_1(h),\sigma) v(h)$$

and observe that v is bounded by $\overline{u}$ so that the final terms disappears asymptotically

Application – job search

three states: unemployed (u), have a bad job (b), have a good job (g)

the only choice: whether or not to quit a bad job and become unemployed

pr(g|g) = 1 (good job is absorbing)

pr(g|u) = a > b = pr(g|b, not quit) (chance of getting a good job)

pr(b|u) = c (chance of getting a bad job when unemployed)

u(g) = d

u(b) = 1

u(u) = 0

procedure: find the value function

$$v(g) = d$$

$$v(u) = (1-\delta)0 + \delta(av(g) + cv(b) + (1-a-c)v(u))$$

$$v(b) = \max\begin{cases} (1-\delta) + \delta(bv(g) + (1-b)v(b)) \\ (1-\delta) + \delta v(u) \end{cases}$$

step 0: substitute out v(g)

$$v(u) = (1-\delta)0 + \delta(ad + cv(b) + (1-a-c)v(u))$$

$$v(b) = \max\begin{cases} (1-\delta) + \delta(bd + (1-b)v(b)) \\ (1-\delta) + \delta v(u) \end{cases}$$

case 1: optimum is to quit a bad job

$$v(b) = (1-\delta) + \delta v(u)$$

substitute:

$$v(u) = \delta(ad + c((1-\delta) + \delta v(u)) + (1-a-c)v(u))$$

$$(1 - \delta(1-a-c) - \delta^2 c)v(u) = \delta ad + \delta(1-\delta)c$$

$$v(u) = \frac{\delta ad + \delta(1-\delta)c}{(1 - \delta(1-a-c) - \delta^2 c)}$$

verify the Bellman equation:

$$(1-\delta)+\delta v(u) \geq (1-\delta)+\delta(bd+(1-b)v(b))$$
$$= (1-\delta)+\delta(bd+(1-b)((1-\delta)+\delta v(u)))$$

$$v(u) \geq \frac{bd+(1-b)(1-\delta)}{1-\delta(1-b)}$$

$$\frac{\delta ad+\delta(1-\delta)c}{(1-\delta(1-a-c)-\delta^2 c)} \geq \frac{bd+(1-b)(1-\delta)}{1-\delta(1-b)}$$

for example when $b=0, c=0, a=1$, $\delta d \geq 1$

Interpersonal Preferences

Experimental results

Roth et al [1991]

US $10.00 stake games, round 10

Second and final round of bargaining game:

Player may take x or reject it and get nothing.

The other player gets $10-x

5 of 27 offers with x>0 are rejected

5 of 14 offers with 5>x>0 are rejected

x	Offers	Rejection Probability
$2.00	1	100%
$3.25	2	50%
$4.00	7	14%
$4.25	1	0%
$4.50	2	100%
$4.75	1	0%
$5.00	13	0%
total	27	