# Problems on Reputation

### *1. Reputation*

A sequence of consumers must choose what product to buy from Gigantic Corporation: a mediocre product or a special improved brand. The mediocre product yields a utility to the consumer of 1 and a profit to Gigantic of 1. The special improved brand yields a utility of 2 and a profit of 2. However, Gigantic has the option of producing a cheap imitation brand that is indistinguishable from the special improved brand. This yields a utility of 0 and a profit of 4. If a consumer buys a special improved brand, he finds out whether or not it is the cheap imitation, and reveals this information to later consumers.

a. Show that there is a sequential equilibrium in which Gigantic produces only cheap imitations and consumers always buy the mediocre product.

b. If Gigantic is very patient and there is a positive probability that it is "honest" and does not produce imitations, does this make a difference?

c. Would it make a difference if Gigantic has also the option of producing defective products that are indistinguishable from mediocre products? These yield a utility of -1 and a profit of 0.

d. What if in part c) all pure strategy types have equal probability?.

### *2. Inference and Martingales*

A single decision-maker picks a sequence of actions $a_t \in A$, a finite set. He is drawn from a finite set of types $\Omega$. If $h_t = (a_1, a_2, \ldots a_t)$ is the history of his play through *t* his strategy may be described by a probability distribution over *A* at time *t*, $\sigma_t(h_{t-1}, \omega)$, which depends on the history and his type. You observe the play of this player, and place probability $\mu(\omega) > 0$ on his being type $\omega$.

Consider $\mu(\omega | h_t)$. By Bayes law

$$\mu(\omega \mid h_t) = \frac{\sigma_t(h_{t-1}, \omega)(a_t)\mu(\omega \mid h_{t-1})}{\sum_{\omega'} \sigma_t(h_{t-1}, \omega')(a_t)\mu(\omega' \mid h_{t-1})}$$

Fix a type $\omega^+$, and let $\Omega^+ \equiv \Omega \setminus \omega^+$ be the set of all other types. We may define random variables $p_t, q_t$ by

$$p_t(a) = \sigma_t(h_{t-1}, \omega^+)(a_t), p_t = p_t(a_t)$$

$$q_t(a) = \frac{\sum_{\omega' \in \Omega^+} \sigma_t(h_{t-1}, \omega')(a_t)\mu(\omega' \mid h_{t-1})}{1 - \mu(\omega^+ \mid h_{t-1})}, q_t = q_t(a_t)$$

We also define $L_t$ recursively by

$$L_0 = \frac{1 - \mu(\omega^+)}{\mu(\omega^+)}$$

$$L_t = \frac{q_t}{p_t} L_{t-1}$$

a. What are $p_t$ and $q_t$ .

b. Show by induction that

$$L_t = \frac{1 - \mu(\omega^+ \mid h_t)}{\mu(\omega^+ h_t)}$$

c. Show that

$$E[L_t \mid L_{t-1}, h_{t-1}, L_{t-2}, h_{t-2}, \ldots \omega^+) \leq L_{t-1}$$

This means (by definition) that $L_t$ is a supermartingale; obviously $L_t \geq 0$.

d. It is known that if $L_t$ is a non-negative supermartingale, with probability one, the sequence ($L_0$,$L_1$,$L_2$,...) converges to a limit. How can you interpret this fact?

## *3. The Chain Store Paradox-Paradox*

Consider the Kreps-Wilson version of the chain store paradox: An entrant may stay out and get nothing (0), or he may enter. If he enters, the incumbent may fight or acquiesce. The entrant gets *b* if the incumbent acquiesces, and *b-1* if he fights, where *0<b<1*. There are two types of incumbent, both receiving *a>1* if there is no entry. If there is a fight, the strong incumbent gets 0 and the weak incumbent gets -1; if a strong incumbent acquiesces he gets -1, a weak incumbent 0.

Only the incumbent knows whether he is weak or strong; it is common knowledge that the entrant a priori believes that he has a π0 chance of facing a strong incumbent. Define

$$\gamma = \frac{p_0}{1 - p_0} \frac{1-b}{b}$$

a. Sketch the extensive form of this game.

b. Define a sequential equilibrium of this game.

c. Show that if $\gamma \neq 1$, there is a unique sequential equilibrium, and that if $\gamma > 1$ entry never occurs, while if $\gamma < 1$ entry always occurs.

d. What are the sequential equilibria if $\gamma = 1$?

e. Now suppose that the incumbent plays a second round against a different entrant who knows the result of the first round. The incumbent's goal is to maximize the sum of his payoffs in the two rounds. Show that if $\gamma > 1$ there is a sequential equilibrium in which the entrant enters on the first round and both types of incumbents acquiesce. Be careful to specify both the equilibrium strategies and beliefs.