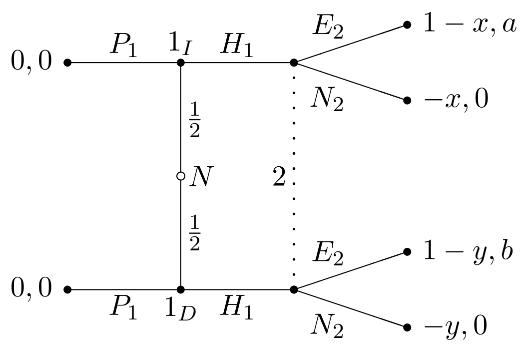
E201B: Final Exam—Suggested Answers

David Rahman, UCLA*,†

Winter, 2003

Sequential Equilibrium and Signalling

We have the following game in extensive form.



We are given the following parametric restrictions: $y > x > 0$, $1 > x$, and $a > 0$.

(*a*) *For what parameter values is there a sequential equilibrium where both types play P?*

For both incarnations of player 1 to play P, player 2 must play N for sure, otherwise player 1_I would want to play H. For player 2 to rationally play N he must believe that $\beta = P(I|H)$ is sufficiently low and $b \leq 0$, otherwise N is strictly dominated. Any β is consistent, and, since player 2's information set is off the equilibrium path, N is immediately sequentially rational.

*Please send any comments to dmr@ucla.edu.

Therefore, the necessary conditions are $b \leq 0$.

(*b*) *For what parameter values is there a separating equilibrium?*

If player 2 plays N, then there is no separating equilibrium, since both incarnations of player 1 will prefer to play P. If player 2 plays E, then there is a separating equilibrium only if $y \geq 1$, so that the dumb player 1, i.e., 1_D, prefers to play P whereas the intelligent player 1, i.e., 1_I, prefers to play H.

Profit Sharing

There are two states of the book's publication: it'll either be a best-seller (B) or a failure (F). The author may either work (W) or shirk (S). The probability of a best-seller depends on the author's effort. Let

$$P(B|W) = H, \qquad P(B|S) = L$$

denote the probability of success given W and S, respectively, where $1 > H > L > 0$. The author's utility function for cash c is given by $\ln(1 + c)$; the cost of effort is $C > 0$.

If the book becomes a best-seller then it'll yield a revenue of $y > 0$, otherwise it'll yield no revenue at all. In case it's optimal to induce high effort, the (risk neutral) publisher's optimization problem is the following.

$$\max_{\theta \in [0,1]} \{(1 - \theta)yH : H \ln(1 + \theta y) - C \geq L \ln(1 + \theta y)\}$$

The Lagrangian for this problem is

$$L = (1 - \theta)yH + \lambda \left[(H - L) \ln(1 + \theta y) - C\right].$$

First-order conditions yield, assuming an interior solution (i.e., $0 < \theta < 1$),

$$\frac{\partial L}{\partial \theta} = -yH + \lambda y \frac{H - L}{1 + \theta y} = 0,$$

which implies that

$$\theta = \frac{1}{y}\left[\lambda \frac{H - L}{H} - 1\right].$$

Therefore, $\lambda > 0$, otherwise θ would necessarily be negative, which is impossible. Since $\lambda > 0$, the incentive constraint must hold with equality, i.e.,

$$(H - L) \ln(1 + \theta y) = C.$$

Rearranging, we obtain that

$$\theta = \frac{1}{y}\left[e^{\frac{C}{H-L}} - 1\right].$$

Using the two expressions for θ we obtain the following value for λ:

$$\lambda = \frac{H}{H-L} e^{\frac{C}{H-L}}.$$

In general (i.e., including corner solutions as well as the possibility that inducing high effort is not optimal), the publisher's optimum θ is

$$\min \left\{ 1, \max \left\{ 0, \frac{1}{y} \left[e^{\frac{C}{H-L}} - 1 \right] \right\} \right\}.$$

Long Run versus Short Run

Consider the following two-player game.

	L	M	R	S
U	$1,1$	$5,4$	$1,\mathbf{5}$	$\mathbf{0},0$
D	$\mathbf{3},\mathbf{5}$	$\mathbf{6},4$	$\mathbf{2},1$	$\mathbf{0},0$

There is one pure NE: (D, L). To find mixed NE, notice that S is strictly dominated, so it won't be played by player 2 in any mixed NE. But then, not only does player 1 have no reason to play U, but also player 1 strictly prefers to play D over U, so there is no mixed Nash equilibrium.

The minmax payoff for player 1 is clearly 0, implemented by player 2 playing S. If player 1 commits to playing U and player 2 best-responds, player 1's payoff will be 1, whereas if player 1 commits to playing D then his payoff will be 3. Hence 3 is player 1's pure Stackelberg precommitment payoff.

For the mixed Stackelberg payoff, let p be the probability with which player 1 plays U. If player 1 commits to the mixed strategy $p < 1/4$ then player 2 will play L, so player 1 will get an expected payoff of $p + 3(1-p)$. If player 1 commits to playing $p > 3/4$ then player 2 will play R, so player 1's payoff will be $p + 2(1-p)$. Finally, if player 1 plays $1/4 \le p \le 3/4$ then player 2 will optimally play M, so that player 1's payoff will be $5p + 6(1-p)$. The highest this could possibly be is when $p = 1/4$, so that the mixed Stackelberg payoff to player 1 is $5/4 + 6(3/4) = 23/4$.

The best dynamic equilibrium payoff for player 1, denoted $\bar{v}_1$, is given by

$$\bar{v}_1 = \max_{(\alpha_1,\alpha_2)} \left\{ \min_{a_1} \{u_1(a_1, \alpha_2) : \alpha_1(a_1) > 0\} : \alpha_2 \in BR_2(\alpha_1) \right\}.$$

If player 1 mixes with probability p such that $1/4 < p < 3/4$ then player 2 will play M whose "worst in support" is 5; this being the best "worst-in-support," it follows that $\bar{v}_1 = 5$.

With Nash threats, the values of $\delta < 1$ that support such payoff are given by

$$5 \geq 6(1-\delta) + 3\delta \qquad \Rightarrow \qquad \delta \geq 1/3.$$

Strategies that support this equilibrium payoff are (I'm assuming that a public randomization device exists):

Device The public randomization device consists of a bent coin (so the state space is $\Omega = \{H, T\}$, with H being heads and T tails) with the following probabilities:

$$P(H) = \frac{1-\delta}{2\delta}, \qquad P(T) = 1 - P(H) = \frac{3\delta - 1}{2\delta}.$$

Player 1

1. Start playing the mixed strategy $\frac{1}{2}[U] + \frac{1}{2}[D]$.
2. If it turns out you played U, then go back to step 1.
3. If it turns out you played D, then use the public randomization device to play $\frac{1}{2}[U] + \frac{1}{2}[D]$ again if H and play D if T.
4. If there ever was a deviation from these strategies or if (D, L) was ever played, then play D forever after, otherwise go back to step 1.

Player 2

1. Start playing the pure strategy M.
2. If it turns out that player 1 played U, then go back to step 1.
3. If it turns out that player 1 played D, then use the public randomization device to play M if H and L if T.
4. If there ever was a deviation from these strategies or if (D, L) was ever played, then play L forever after, otherwise go back to step 1.

Notice that if (and only if) $\delta \geq 1/3$ then the numbers $P(H)$ and $P(T)$ *are* probabilities, i.e., $0 \leq P(H) \leq 1$. It remains to show first that the payoff to player 1 from this strategy profile, call it v_1, is actually equal to $\bar{v}_1$, and that such strategies are equilibrium strategies. To find v_1, notice that, if both players play according to the strategies above, then v_1 must satisfy

$$v_1 = \tfrac{1}{2}[(1-\delta)5 + \delta v_1] + \tfrac{1}{2}[(1-\delta)6 + \delta(3P(H) + v_1 P(T))],$$

where the first term, $\frac{1}{2}[(1-\delta)5 + \delta v_1]$, represents the lifetime payoff to player 1 if it turns out that he played U in the first period, and the second term, $\frac{1}{2}[(1-\delta)6 + \delta(3P(H) + v_1 P(T))]$, represents the payoff to player 1 if it turns out that he played D in the first period. Rearranging, we get (you guessed it) what we wanted, namely that

$$v_1 = \frac{11(1-\delta) + 3\delta P(H)}{2 - \delta(1 + P(T))} = \frac{(1-\delta)(11 + 3/2)}{2 - (5\delta - 1)/2} = \frac{(1-\delta)25/2}{5/2 - 5\delta/2} = 5.$$

To show that this constitutes a dynamic equilibrium, notice that player 2 is always best-responding to player 1, so he has no incentive to deviate. As far as player 1 is concerned there are to phases of play: either his opponent is meant to play M or he's meant to play L. If player 2 is meant to play M, then player 1's (dynamic) payoff if playing U is 5 (since $(1-\delta)5+\delta v_1 = (1-\delta)5+\delta 5 = 5$) and his dynamic payoff if playing U is 5, too, since

$$(1-\delta)6+\delta(3P(H)+v_1P(T)) = (1-\delta)6+\delta\left[3\frac{1-\delta}{2\delta}+5\frac{3\delta-1}{2\delta}\right] = 5.$$

Therefore, player 1 has no incentive to deviate *on the equilibrium path*. The last thing we need to check is that player 1 doesn't want to deviate from the proposed strategies when player 2 is meant to play L. But since player 1 is meant to play D, this is just the static Nash equilibrium forever after, so trivially there's no incentive to deviate.

Remark *Let me mention that the strategies above are special in the following way. First, we calculated the values of δ for which the best dynamic equilibrium payoff was sustainable and then we designed the strategies. But* equivalently, *we could have begun with the strategies and asked, for what values of δ would the strategies be well-defined? I.e., for what values of δ is it the case that $0 \leq P(H) \leq 1$? Indeed, we obtain that $\delta \geq 1/3$. This two-way street is a general principle. (See Fudenberg, Levine, and Maskin, 1989.) The trick in concocting the strategies was to choose $P(H)$ such that the dynamic payoff to player 1 is the same regardless of the pure strategy played amongst the strategies with positive probability.*

Now let's find the worst dynamic equilibrium payoff for player 1, call it $\underline{v}_1$. This is given by the "constrained minmax," i.e.,

$$\underline{v}_1 = \min_{(\alpha_1,\alpha_2)}\left\{\max_{b_1}\{u_1(b_1,\alpha_2)\} : \alpha_2 \in BR_2(\alpha_1)\right\}.$$

For this game, it is clear that $\underline{v}_1 = 2$. This is because the constraint restricts player 2 not to play S, leading to the worst possible best response for player 1 being 2.

To find the critical value of $\delta < 1$ above which (yes, above which) $\underline{v}_1$ is attainable as a dynamic equilibrium payoff for player 1, note that by definition of $\underline{v}_1$, it must satisfy the following condition:

$$\underline{v}_1 = 1(1-\delta)+w_1(U)\delta, \qquad w_1(U) \leq 3;$$

the first equation follows because we cannot possibly have (D,R) being played, since R is not a best response to D and player 2 *must* play a static best response by virtue of being myopic. It follows that, for it to be possible that player 1 gets a lifetime utility of 2, player 1 must play U (and player 2 must play R) for some time. The second expression is the "Nash threats" condition. Plugging the second expression into the first one, we obtain

$$\begin{aligned}\underline{v}_1 &= 1(1-\delta)+w(U)\delta \\ &\leq 1(1-\delta)+3\delta \\ &\leq 1+2\delta,\end{aligned}$$

which after substituting for $\underline{v}_1 = 2$ and rearranging to bound δ yields

$$\delta \geq \frac{\underline{v}_1 - 1}{2} = \frac{1}{2}.$$

For strategies that implement such a payoff assuming ($\delta \geq 1/2$), consider the following. Note that the same remark applies as for the calculation of the critical δ associated with $\bar{v}_1$.

Device The public randomization device consists of a bent coin (so the state space is $\Omega = \{H, T\}$, with H being heads and T tails) with the following probabilities:

$$P(H) = \frac{2\delta - 1}{\delta}, \qquad P(T) = 1 - P(H) = \frac{1 - \delta}{\delta}.$$

Player 1

1. Start playing the pure strategy U.
2. If there were no deviations by any player so far, then use the public randomization device to go back to step 1 if H and play D forever after if T.
3. If there ever was a deviation from these strategies then go back to step 1.

Player 2

1. Start playing the mixed strategy R.
2. If there were no deviations by any player so far, then use the public randomization device to go back to step 1 if H and L forever after if T.
3. If there ever was a deviation from these strategies then go back to step 1.

Now we need to show that player 1 gets 2 utils from this profile and that it's an equilibrium. Indeed, player 1's payoff from this profile, call it v_1, is given by

$$v_1 = 1(1 - \delta) + \delta \left[v_1 P(H) + 3P(T) \right],$$

which, after solving for v_1, it turns out that

$$v_1 = \frac{(1 - \delta) + 3\delta P(T)}{1 - \delta P(H)} = \frac{(1 - \delta) + 3(1 - \delta)}{1 - (2\delta - 1)} = \frac{4(1 - \delta)}{2(1 - \delta)} = 2,$$

as required. To show that this is a dynamic equilibrium, notice first of all that player 2 is always best-responding, so he has no incentive to deviate. For player 1, clearly he won't deviate at the phase where players play the static Nash (D, L). I claim that player 1 won't deviate when he's meant to play U. Indeed,

$$2 \geq (1 - \delta)2 + \delta v_1 = 2,$$

where the left-hand side is the payoff from playing the prescribed strategy and the right-hand side is the best payoff associated with a deviation. (I.e., playing D instead of U in the current period and then reverting to the original strategies for the subsequent periods. That this is without loss of generality is known as the *one-shot deviation principle*, see Fudenberg and Tirole or Myerson (page 319, Theorem 7.1 and subsequent discussion.) We're done, as there's no other opportunity for player 1 to deviate. The key point in all this was that player 1 is punished by *delaying* the reversion to the static Nash.