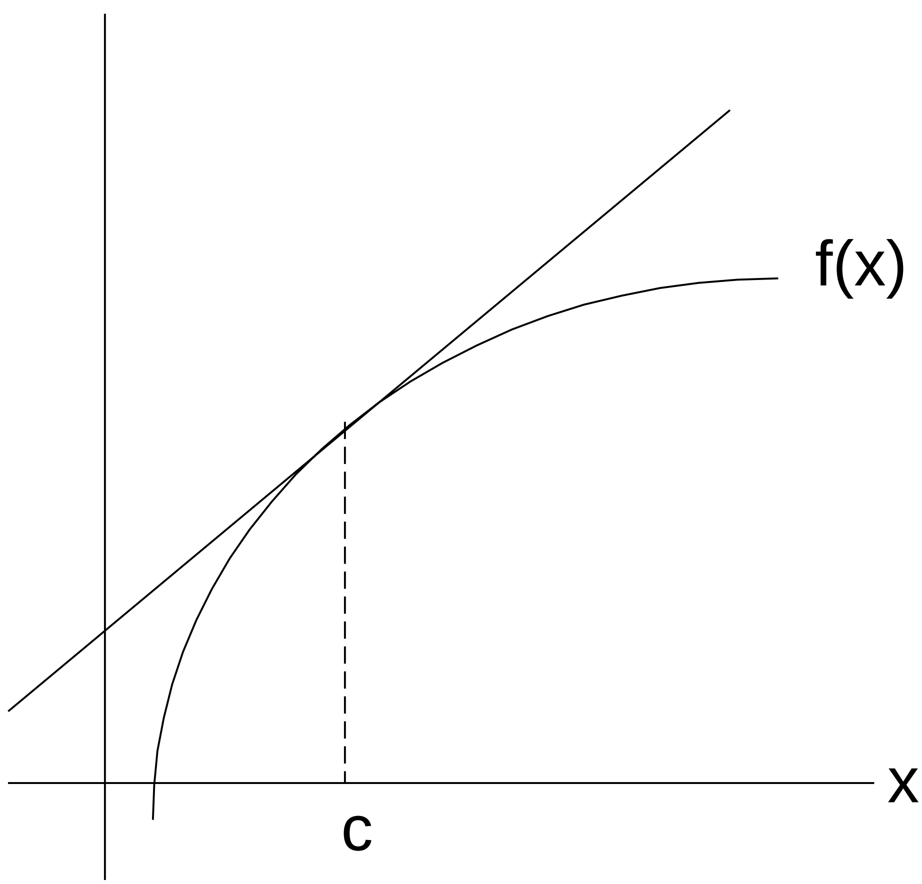
Lagrange Multipliers

$\max f(x)$ subject to $x \leq c$

obvious answer: if f is concave and $f'(c) > 0$ take $x = c$



Lagrange Multipliers

$$L = f(x) - \lambda(x - c)$$

first order conditions for a maximum

$$\frac{\partial L}{\partial x} = f'(x) - \lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = -(x - c) = 0$$

solution: $x = c, \lambda = f'(x)$

what is measured by the multiplier λ?
the increase in the objective when the constraint *c* is relaxed

Example

max $\left(x_1{}^{\rho} + x_2{}^{\rho} + x_3{}^{\rho}\right)^{1/\rho}$
subject to $p_1 x_1 + p_2 x_2 + p_3 x_3 = I$

Lagrangean

$$L = \left(x_1{}^{\rho} + x_2{}^{\rho} + x_3{}^{\rho}\right)^{1/\rho} - \lambda(p_1 x_1 + p_2 x_2 + p_3 x_3 - I)$$

solution:

$$\frac{\partial L}{\partial x_i} = \frac{1}{\rho}\left(x_1{}^{\rho} + x_2{}^{\rho} + x_3{}^{\rho}\right)^{(1/\rho)-1} \rho x_i{}^{\rho-1} - \lambda p_i = 0$$

solve:

$$x_i = \left[\frac{\lambda p_i}{\left(x_1{}^{\rho} + x_2{}^{\rho} + x_3{}^{\rho}\right)^{(1/\rho)-1}}\right]^{\frac{1}{\rho-1}}$$

$$= \left[\frac{p_i}{\left(x_1{}^{\rho} + x_2{}^{\rho} + x_3{}^{\rho}\right)^{(1/\rho)-1}}\right]^{\frac{1}{\rho-1}} [\lambda]^{\frac{1}{\rho-1}}$$

substitute in constraint

$$\sum_i p_i^{1+\frac{1}{\rho-1}} \left[\frac{\lambda}{\left(x_1{}^{\rho} + x_2{}^{\rho} + x_3{}^{\rho}\right)^{(1/\rho)-1}} \right]^{\frac{1}{\rho-1}} = I$$

$$[\lambda]^{\frac{1}{\rho-1}} = \frac{I}{\sum_i p_i^{\frac{\rho}{\rho-1}} \left(x_1{}^{\rho} + x_2{}^{\rho} + x_3{}^{\rho}\right)^{\frac{1}{\rho}}}$$

substitute back into solution of FOC

$$x_i = \left[\frac{p_i}{\left(x_1{}^{\rho} + x_2{}^{\rho} + x_3{}^{\rho}\right)^{(1/\rho)-1}} \right]^{\frac{1}{\rho-1}} \frac{I}{\sum_j p_j^{\frac{\rho}{\rho-1}} \left(x_1{}^{\rho} + x_2{}^{\rho} + x_3{}^{\rho}\right)^{\frac{1}{\rho}}}$$

$$= [p_i]^{\frac{1}{\rho-1}} \frac{I}{\sum_j p_j^{\frac{\rho}{\rho-1}}}$$

this is the *CES* demand function widely used in empirical research

note that it satisfies homogeneity of degree zero in prices and income