Solutions Key to
Problem Sets in Game Theory

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Problem Set 1: Problems on Static Games

Exercise 1: Dominance and Equilibrium

For each of the following games find:

1) *all weak and strict dominant strategy equilibria (WDS/ SDS)*
2) *apply iterated strict dominance (ISD)*
3) *find all pure and mixed Nash equilibria (PSNE, MSNE)*
4) *indicate which NE are trembling hand perfect (THP) and why.*

a) Consider the following game:

		P2 L	P2 R
P1	U	2,1	0,0
P1	D	0,0	1,2

1) We first look at the best responses of each player.

$$BR_1(L) = U \qquad BR_1(R) = D$$
$$BR_2(U) = L \qquad BR_2(D) = R$$

No player has a weakly or strictly dominant strategy. Hence, there is no pure strategy equilibrium in strictly or weakly dominant strategies. There is no mixed strategy equilibrium in weakly or strictly dominant strategies either. To see this, take any convex combination of U-D for P2: it will pay less than just playing U. The same applies to P1.

*This version builds on the solutions provided by Damiano Argan and Konuray Mutluer.

2) It is not possible to apply ISD since players have no dominant strategies.

3) PSNE: Based on the BRs, there are two

$$PSNE = \{(U, L), (D, R)\}$$

MSNE: Call σ_L the probability that P2 always plays L (and equivalently $\sigma_R = 1 - \sigma_L$ for R). P2 will set his probability to make P1 indifferent between playing his strategies.

$$2\sigma_L = 1 - \sigma_L \qquad \sigma_L = 1/3 \qquad \sigma_R = 2/3$$

By symmetry, we know $\sigma_U = 2/3$ and $\sigma_D = 1/3$.

Always remember to include the PSNE of the game when writing the MSNE.

$$MSNE = \left\{ \left(\frac{2}{3}U + \frac{1}{3}D, \frac{1}{3}L + \frac{2}{3}R\right), \Big(1U + 0D, 1L + 0R\Big), \Big(0U + 1D, 0L + 1R\Big) \right\}$$
$$MSNE = \big\{ \big(\sigma_U = 2/3, \sigma_L = 1/3\big), \big(\sigma_U = 1, \sigma_L = 1\big), \big(\sigma_U = 0, \sigma_L = 0\big) \big\}$$

4) All the equilibria are trembling hand perfect.

Proposition. Trembling hand perfect (THP)
The strategy $\sigma = (\sigma_1, \ldots, \sigma_N)$ is THP in a two player game if it does not attach any probability to weakly dominated strategies.

In our PSNE, any deviation yields strictly lower payoffs.

Proof. Consider the PSNE $\{U, L\}$ and add a slight tremble $\varepsilon_n = 1/3n$ (now it is a fully mixed sequence!).

$$\sigma^n = \{\sigma_U^n, \sigma_L^n\} = \left\{ \left(1 - \frac{1}{3n}\right), \left(1 - \frac{1}{3n}\right) \right\}$$

The sequence converges to the PSNE $\{U, L\}$ as $n \to \infty$. In this case, the BR are: [1]

$$\left. \begin{aligned} BR_1(\sigma_L^n) = U \\ BR_2(\sigma_U^n) = L \end{aligned} \right\} \forall n$$

□

The MSNE contains a fully mixed strategy, since both PSNE are played with strictly positive probability. So we can take the fully mixed sequence sequence $\sigma^n = \sigma^{MSNE} \forall n$. This (constant) sequence trivially converges to σ^{MSNE}, yielding the same BR.

b) Consider the game:

1) D is a SDS for P1 as $BR_1(L) = BR_1(R) = D$.

R is a SDS for P2 as $BR_2(U) = BR_2(D) = R$.

The strategy $\sigma^D = \{(D, R)\}$ constitutes a strictly dominant strategy equilibrium.

[1] $u(U, \sigma_L^n) = 2 - \dfrac{2}{3n} > \dfrac{1}{3n} = u(D, \sigma_L^n)$

P2

		L	R
P1	U	6,6	0,7
	D	7,0	1,1

2) There is only one strategy profile surviving ISD. This game is called dominance solvable, and any dominance solvable game has a unique NE.

   Start with P1: U is strictly dominated by D, hence we remove it. As for P2, L is strictly dominated by R for P2, hence we remove it.

3) Therefore, the unique (pure and mixed) NE is $\{(D, R)\}$. There is no **totally** MSNE as no strictly dominated strategy is played with positive probability in a NE.

4) Any deviation from this unique equilibrium yields strictly lower payoffs. [2] Hence, this is a strict NE and in any two player game a strict NE is THP.

c) Consider the game:

P2

		L	C	R
P1	U	3,3	2,2	1,1
	M	2,2	1,1	0,8
	D	1,1	8,0	0,0

1) We first look at the best responses of each player.

$$BR_1(L) = U \qquad BR_1(C) = D$$
$$BR_2(U) = L \qquad BR_2(M) = R$$

   There are no strict nor weak dominant strategies for P1 nor P2.

2) We start with P1: M is strictly dominated by U, we remove it. Move to P2: C is strictly dominated by L, remove it. Then R is strictly dominated by L, so remove it. Back to P1, D is strictly dominated by U, remove it.

   The iterated removal of strict dominant strategy leads to $\{(U, L)\}$ as the unique ISD profile.

3) The game is dominance solvable. The only NE of the game is $\{(U, L)\}$. There is no pure MSNE because in a NE no strictly dominated strategies are played with positive probability.

4) Since the strategy profile in the NE does not involve a weakly dominated strategy, it is THP.

d) Consider the game:

[2] Alternatively, no weakly dominated strategy is involved.

		P2	
		L	R
P1	U	1,3	1,3
	D	0,0	2,0

1) We first look at the BRs of each player:

$$BR_1(L) = U \qquad BR_1(R) = D$$
$$BR_2(U) = \{L, R\} \qquad BR_2(D) = \{L, R\}$$

No strategy is strictly nor weakly dominant for P1.

Note that $u_2(\sigma_1, L) = u_2(\sigma_1, R)\ \forall \sigma_1 \in \Delta\{U, D\}$. P2 is indifferent between all of her strategies given σ_1. Remember that for a strategy to be weakly dominant it must performs as good as the others for whatever strategy the other player could play **and better for at least one strategy.** This last part is not satisfied in this example, hence there are no strictly nor weakly dominant strategies for P2.

As neither player has a strictly dominant strategy, the game does not yield any dominant strategy equilibria.

2) Iterated elimination is not possible, as there are no strictly dominant strategies.

3) Remark that P2 is always completely indifferent between L, R (they deliver the same payoffs), so assigning <u>**any**</u> probability to them will be a a BR.

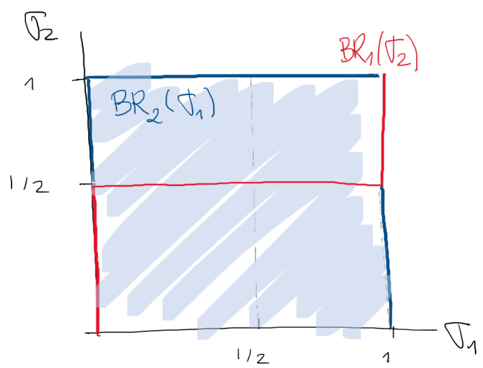
$$BR_2(\sigma_1) = \Delta\{L, R\}$$

P1, however, is not always indifferent between P2's strategies. Let's look at the indifference condition for P1.

$$1\sigma_L + 1(1 - \sigma_L) = 0\sigma_L + 2(1 - \sigma_L) \rightarrow \sigma_L = \frac{1}{2}$$

$$BR_1(\sigma_2) = \begin{cases} D & \text{if } \sigma_L < 1/2 \\ \Delta\{U, D\} & \text{if } \sigma_L = 1/2 \\ U & \text{if } \sigma_L > 1/2 \end{cases}$$

Graphically, the BR are:



$$MSNE = \left\{ \left(\sigma_U = 0, \sigma_L < \frac{1}{2} \right); \left(0 \le \sigma_U \le 1, \sigma_L = \frac{1}{2} \right); \left(\sigma_U = 1, \sigma_L > \frac{1}{2} \right) \right\}$$

$$PSNE = \{(\sigma_U = 1, \sigma_L = 1), (\sigma_U = 0, \sigma_L = 0)\}$$

4) No weakly dominant strategy is involved in any of the equilibria: all equilibria are THP.

Let's look closer at the subsets of MSNE. We will perturb the strategies by adding an error ε_n in each strategy profile. This error will depend on the equilibrium strategy:

- Consider $\sigma^\star = (\sigma_U^\star = 1, \sigma_L^\star > 1/2)$.
  We can convert it into the fully mixed sequence $\sigma^n = (\sigma_U^n, \sigma_L^n) = \{1 - \varepsilon_n, \sigma_L^\star - \varepsilon_n\}$. Let's give a closer look at how we choose ε_n. We need that $\lim_{n\to\infty} \sigma^n = \sigma^\star$, and for this it must be that $\lim_{n\to\infty} \varepsilon_n = 0$. These two conditions give us bounds for the values of the "tremble": $\varepsilon_n \in (0, \sigma_L^\star - 1/2) \forall n$. A candidate would be:

  $$\varepsilon_n = \frac{\sigma_L^\star - 1/2}{2n}$$

  $$\sigma^n = (\sigma_U^n, \sigma_L^n) = \{1 - \varepsilon_n, \sigma_L^\star - \varepsilon_n\} = \left\{ 1 - \frac{\sigma_L^\star - 1/2}{2n}, \sigma_L^\star - \frac{\sigma_L^\star - 1/2}{2n} \right\}$$

  Since $\sigma_L^n \ge 1/2 \forall n$, U is always a BR for P1 to σ_L^n. [3] Any strategy is a BR for P2. Hence, these equilibria are THP.
- Consider $\sigma^\star = (\sigma_U^\star = 1, \sigma_L^\star = 1/2)$.
  We can convert it into the fully mixed sequence $\sigma^n = (\sigma_U^n, \sigma_L^n) = \{1 - \varepsilon_n, \sigma_L\}$. Again, we need that $\lim_{n\to\infty} \sigma^n = \sigma^\star$, and for this it must be that $\lim_{n\to\infty} \varepsilon_n = 0$. These two conditions give us bounds for the values of the "tremble": $\varepsilon_n \in (0, 1/2)$. Choose, for example, $\varepsilon_n = 1/3n$. Since $\sigma_L^n = 1/2 \forall n$, U is always a BR for P1 to σ_L^n (and any strategy is a BR for P2 to any strategy played by P1). Hence, these equilibria are THP.
- Consider $\sigma^\star = (\sigma_U^\star = 0, \sigma_L^\star = 1/2)$, and modify is so that $\sigma^n = \{\sigma_U^n, \sigma_L^n\} = \{\varepsilon_n, \sigma_L^\star\}$. Once again, we need that $\lim_{n\to\infty} \sigma^n = \sigma^\star$, and for this it must be that $\lim_{n\to\infty} \varepsilon_n = 0$. These two conditions give us bounds for the values of the "tremble": $\varepsilon_n \in (0, 1/2)$. Choose, for example, $\varepsilon_n = 1/3n$. Since $\sigma_L^n = 1/2 \forall n$, D is always a BR for P1 to σ_L^n (and any strategy is a BR for P2 to any strategy played by P1). Hence, these equilibria are THP.
- Consider $\sigma^\star = (\sigma_U^\star = 0, \sigma_L^\star < 1/2)$, and modify is so that $\sigma^n = \{\sigma_U^n, \sigma_L^n\} = \{\varepsilon_n, \sigma_L^\star + \varepsilon_n\}$. Once more, we need that $\lim_{n\to\infty} \sigma^n = \sigma^\star$, and for this it must be that $\lim_{n\to\infty} \varepsilon_n = 0$. These two conditions give us bounds for the values of the "tremble": $\varepsilon_n \in (0, \sigma_L^\star) \forall n$. Choose, for example, $\varepsilon_n = \dfrac{1/2 - \sigma_L^\star}{2n}$. Since $\sigma_L^n \le 1/2 \forall n$, D is always a BR for P1 to σ_L^n. Any strategy is a BR for P2. These equilibria are THP, as $\sigma^n \to \sigma^\star$.
- Consider $\sigma^\star = (0 < \sigma_U^\star < 1, \sigma_L^\star = 1/2)$. These equilibria are fully mixed, which means that they are assigning a strictly positive probability to every strategy in the game. This is exactly the definition of THP, so they are automatically THP.

[3] If you are not convinced yet, check that $u(U, \sigma_L^n) > u(D, \sigma_L^n) \forall n$

Exercise 2: Dominance and Nash Equilibrium

Prove that a profile $\sigma^\star = (\sigma_1, \dots \sigma_I)$ is a Nash equilibrium of a game Γ if and only if it is the Nash equilibrium of the game in which strategies have been removed by iterated strict dominance Γ^N.

Some notation: we are calling Γ the collection of strategies at the original game, Γ^N the collection of strategies of the reduced form, and Γ_i^n the collection of strategies available for player i at iteration $n \in \mathbb{N}$.

WTS: $\sigma^\star$ NE in $\Gamma \quad \longleftrightarrow \quad \sigma^\star$ NE in Γ^N. We proceed to show each implication in turns.

<u>$\sigma^\star$ NE in $\Gamma \quad \longrightarrow \quad \sigma^\star$ NE in Γ^N</u>

Two-step proof. For this implication, we need to show two things:

1. $\sigma^\star$ is a NE in Γ **then** $\sigma^\star \in \Gamma^N$, ie, a NE profile cannot be eliminated by IESDS.

   Proof by contradiction.
   Suppose $\sigma^\star \notin \Gamma^N$. If this is the case, some strategy $\sigma_i^\star$ from $\sigma^\star$ has to be eliminated at an iteration of the game Γ^n with $n < N$. Call σ_i' the strategy that strictly dominates $\sigma_i^\star$. The definition of strict dominance implies that:

   $$u\left(\sigma_i', \sigma_{-i}^\star\right) > u\left(\sigma_i^\star, \sigma_{-i}^\star\right)$$

   This would constitute a profitable deviation from $\sigma_i^\star$, which contradicts $\sigma^\star$ being a NE of the initial game.

2. $\sigma^\star$ is a NE in Γ **and** $\sigma^\star \in \Gamma^N \quad \rightarrow \quad \sigma^\star$ is a NE in Γ^N.

   Direct proof.
   Since $\Gamma^N \subseteq \Gamma$, and there are no profitable deviations from $\sigma^\star$ in Γ, there cannot be any profitable deviations in any subset of Γ.

$\square$

<u>$\sigma^\star$ NE in $\Gamma^N \quad \longrightarrow \quad \sigma^\star$ NE in Γ</u>

Proof by contradiction. Suppose $\sigma^\star$ is a NE of Γ^N but not of Γ. Thus, there exists $\sigma_i' \in \Gamma_i$ and $\sigma_i' \notin \Gamma^N$ that constitutes a profitable deviation:

$$u\left(\sigma_i', \sigma_{-i}^\star\right) > u\left(\sigma_i^\star, \sigma_{-i}^\star\right)$$

But since $\sigma_i' \notin \Gamma^N$, at some point it must be dominated by a strategy $\sigma_i'' \in \Gamma_i^n$ with $n < N$. As the strategies of opponents survive ISD, $\sigma_{-i}^\star \in \Gamma_{-i}^N$ means $\sigma_{-i}^\star \in \Gamma_{-i}^n$, the strategy σ_i'' at stage n implies:

$$u\left(\sigma_i'', \sigma_{-i}^\star\right) > u\left(\sigma_i', \sigma_{-i}^\star\right)$$

Unless $\sigma_i'' \in \Gamma_i^N$, the same will apply to σ_i'' with other strategies until we reach the N^{th} iteration $\tilde{\sigma}_i \in \Gamma^N$. Players have a finite number of strategies, so we only need to repeat this argument $N-n$ times.

$$u\left(\tilde{\sigma}_i, \sigma_{-i}^{\star}\right) > u\left(\sigma_i'', \sigma_{-i}^{\star}\right)$$

Since $\sigma^{\star}$ is a NE of Γ^N, it must be weakly better off for i than any other strategy in Γ_i^N:

$$u\left(\sigma_i^{\star}, \sigma_{-i}^{\star}\right) \geq u\left(\tilde{\sigma}_i, \sigma_{-i}^{\star}\right) > u\left(\sigma_i'', \sigma_{-i}^{\star}\right) > u\left(\sigma_i', \sigma_{-i}^{\star}\right) > u\left(\sigma_i^{\star}, \sigma_{-i}^{\star}\right)$$

which cannot be (a contradiction). A strictly dominated strategy will never be a NE. □

Prove that a Nash equilibrium of a game in which strategies have been removed by iterated weak dominance is a Nash equilibrium of the original game.

WTS: $\sigma^{\star}$ is a NE in $\Gamma^{NW} \quad \longrightarrow \quad \sigma^{\star}$ is a NE in Γ

Proof by (sequential) induction. We will work with a sequence of games, starting with the reduced game where all weakly dominated strategies have been eliminated, and will assign names to the games that result from introducing one (and only one) weakly dominated strategy for each player at a time.

- Γ^k is the game at iteration k. I also refer to Γ_i^k as the set of strategies available for player i at game Γ^k.
- σ_i^k is a weakly dominated strategy for player i, the elimination of which produces the game Γ^k.
- Γ^{k+1} is the game resulting from the introduction of σ_i^k in Γ^k. The full game with all the strategies is Γ^K.

Using this notation, we can also say that Γ^k is Γ^{k+1} without σ_i^k.

Inductive step:
Let me first prove that, if $\sigma^{\star}$ is a NE at Γ^k, then it is a NE at Γ^{k+1} $\forall k$. [4]

Inductive hypothesis: Consider the game Γ^k, and suppose that this game has a Nash equilibrium $\sigma^{\star}$. By definition:

$$u(\sigma_i^{\star}, \sigma_{-i}^{\star}) \geq u(\sigma_i', \sigma_{-i}^{\star}) \quad \forall \sigma_i' \in \Gamma_i^k$$

[4]The induction principle consists of:

1. A base case: A(1) is true
2. An inductive step: for all $n \in \mathbb{N}$, the following implication is true: if $A(n)$ is true (inductive hypothesis), then $A(n+1)$ is also true (inductive claim).

Then $A(n)$ is true for all $n \in \mathbb{N}$.

Let's add a weakly dominated strategy σ_i^k to Γ^k, so that we find ourselves in the game Γ^{k+1}. The *inductive claim* we need to check is that, by moving from Γ^k to Γ^{k+1}, $\sigma^\star$ remains a NE.

As σ_i^k is a weakly dominated strategy, we know that, for some $\sigma_i' \in \Gamma_i^{k+1}$:

$$u(\sigma_i^k, \sigma_{-i}) \leq u(\sigma_i', \sigma_{-i}) \quad \forall \sigma_{-i} \in \Gamma_{-i}^{k+1}$$

In particular, we can say that:

$$u(\sigma_i^k, \sigma_{-i}^\star) \leq u(\sigma_i', \sigma_{-i}^\star)$$

for some $\sigma_i' \in \Gamma_i^{k+1}$. Note that these $\sigma_i' \in \Gamma_i^{k+1}$ are actually the strategies present in Γ^k. Hence, we have:

$$u(\sigma_i^k, \sigma_{-i}^\star) \leq \underbrace{u(\sigma_i', \sigma_{-i}^\star) \leq u(\sigma_i^\star, \sigma_{-i}^\star)}_{\text{From } \sigma^\star \text{ NE in } \Gamma^k}$$

And we can now claim that, in game Γ^{k+1}, we have:

$$u(\sigma_i^\star, \sigma_{-i}^\star) \geq u(\sigma_i', \sigma_{-i}^\star) \quad \forall \sigma_i' \in \Gamma_i^{k+1}$$

By moving from game Γ^k to game Γ^{k+1} the strategy profile $\sigma^\star$ remains a NE.

$$\sigma^\star \text{ is a NE in } \Gamma^k \quad \longrightarrow \quad \sigma^\star \text{ is a NE in } \Gamma^{k+1} \quad \forall k \tag{1}$$

Base case:
Let's now turn to Γ^1, the final game without any weakly dominated strategies. The statement tells us that there exists a NE in Γ^1, so that:

$$u(\sigma_i^\star, \sigma_{-i}^\star) \geq u(\sigma_i', \sigma_{-i}^\star) \quad \forall \sigma_i' \in \Gamma_i^1$$

Therefore, we can apply (1) and claim that the game Γ^2, resulting from adding a weakly dominated strategy will also have $\sigma^\star$ as a NE. We can iterate this result as many times as needed (finitely many times, since the number of strategies is finite). In each iteration we include one weakly dominated strategy for each player, until we reach the original game Γ^K, so that:

$$\sigma^\star \text{ is a NE in } \Gamma^1 \quad \longrightarrow \quad \sigma^\star \text{ is a NE in } \Gamma^K$$

$\square$

Importantly, the opposite implication is not true! $\sigma^\star$ is a NE in $\Gamma \quad \nrightarrow \quad \sigma^\star$ is a NE in Γ^{NW}. A NE of the original game can be removed by weak iteration.

Give an example of a Nash equilibrium of a game that is not a Nash equilibrium of the game where strategies have been removed by iterated weak dominance.

$$PSNE = \{(D, L), (D, R)\}$$

Both U and M are weakly dominated by D. If we first eliminate strategy U, we then eliminate strategy L and we can then eliminate strategy M, yielding as prediction (D, R). If we change the order of elimination, we could end up with (D, L) as a prediction.

		P2	
		L	**R**
	U	5,1	4,0
P1	**M**	6,0	3,1
	D	6,4	4,4

Exercise 3: Correlated Equilibrium

Consider the game depicted below. Show that the proposed correlated strategy profile is in fact a correlated equilibrium.

		P2	
		L	R
P1	U	0,0	2,1
	D	1,2	0,0

		P2	
		L	R
P1	U	$\frac{1}{3}$	$\frac{1}{3}$
	D	$\frac{1}{3}$	-

To define a correlated equilibrium you have to define the probability space, the partitions and the strategies for each player. We proceed as follows:

1. Define the set of outcomes Ω and its associated probability measure (the probabilities over the outcomes we want to induce).

$$\Omega = \{UL, UR, DL, DR\}$$
$$\pi(UL) = \pi(UR) = \pi(DL) = \frac{1}{3} \qquad \pi(DR) = 0$$

2. Define the partition over outcomes so that each player cannot distinguish between states for the world that have the same action for her:

$$P_1 = \{\{UL, UR\}, \{DL, DR\}\}$$
$$P_2 = \{\{UL, DL\}, \{UR, DR\}\}$$

3. Define the strategies each player will play upon receiving the message $\sigma = (\sigma_1, \sigma_2)$:

$$\sigma_1(UL) = \sigma_1(UR) = U \qquad \sigma_1(DL) = \sigma_1(DR) = D$$
$$\sigma_2(UL) = \sigma_2(DL) = L \qquad \sigma_2(UR) = \sigma_2(DR) = R$$

We will now check that this is a correlated equilibrium that induces the probabilities over the outcome that we are looking for. To show that $\sigma = (\sigma_1, \sigma_2)$ is a correlated equilibrium we have to show that $\sigma_1 \in BR_1(\sigma_2)$ and $\sigma_2 \in BR_2(\sigma_1)$.

- The correlated device draws $\{U, L\}$.
  For P1, he assigns the same probability to $\{U, L\}$ and $\{U, R\}$. He believes that P2 is ran-

domizing with probability 1/2. [5] If he plays U, he gets $0 \times 1/2 + 2 \times 1/2 = 1$, whereas if he plays D he expects to get $1 \times 1/2 + 0 \times 1/2 = 1/2$. Therefore, $BR_1(UL) = U$.
Similarly, for P2, his expected payoffs from playing L are $0 \times 1/2 + 2 \times 1/2 = 1/2$, which is greater than his expected payoff from playing R $1 \times 1/2 + 0 \times 1/2 = 1/2$. Hence $BR_2(UL) = L$.

- The correlated device shows $\{U, R\}$.
  The $BR_1(UR) = U$. For P2, playing R yields higher profits than L, so $BR_2(UR) = R$.

- The correlated device shows $\{D, L\}$.
  For P1, he will recognize that P2 will play L with probability 1, and hence $BR_1(DL) = D$. P2, however, believes 1 is randomizing 50/50, so that his payoff from playing L is greater than that of playing R. $BR_2(DL) = L$.

- The correlated device will draw $\{D, R\}$ with probability zero, so we do not need to worry about this case.

When UL is drawn equilibrium strategies of players induce (U,L), when UR is drawn they induce (U,R), when DL is drawn they induce (D,L), and they do so one third of the time each. Therefore, $[(\Omega, \pi), P, \sigma]$ as above defined is a correlated equilibrium.

The payoff profile induced by this equilibrium is (1,1):

$$u_1(\cdot) = \frac{1}{3} \times 0 + \frac{1}{3} \times 2 + \frac{1}{3} \times 1 = 1$$
$$u_2(\cdot) = \frac{1}{3} \times 0 + \frac{1}{3} \times 1 + \frac{1}{3} \times 2 = 1$$

Exercise 4: Anti-Coordination

Two players must choose whether to specialize – they must choose between being a hunter and a gatherer. After they choose, they meet to play a game. If both are hunters, or both are gatherers, they get no benefit from specialization, and receive a utility of zero. If one is a hunter and one a gatherer, the hunter receives 2 and the gatherer 1 unit of utility.

1. *Write the normal form of the game.*

P2

		H	G
P1	H	0,0	2,1
	G	1,2	0,0

[5] This 1/2 probability comes from Bayes rule:

$$\Pr(UL|U) = \frac{\Pr(UL|U)\Pr(UL)}{\Pr(U)} = \frac{1/3}{1/3 + 1/3} = \frac{1}{2}$$

2. *Find the symmetric Nash equilibrium in which both players employ the same strategy.*

   The $PSNE = \{(G, H), (H, G)\}$ is not symmetric. Hence, we move to mixed strategies.

   Consider P2 plays H with probability p to make P1 indifferent between his options:

$$0p + 2(1-p) = p + 0(1-p) \quad \leftrightarrow \quad p = 2/3$$

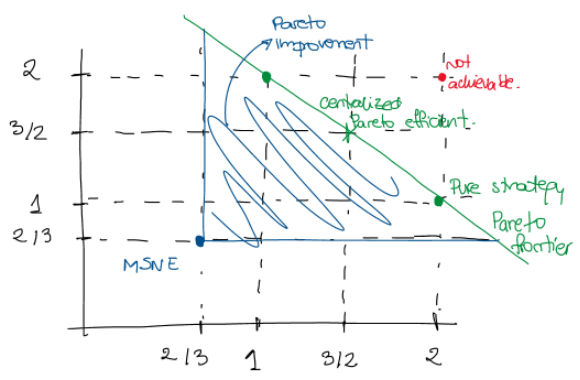
$$\sigma_1(H) = 2/3 \qquad \sigma_1(G) = 1/3$$
$$\sigma_2(H) = 2/3 \qquad \sigma_2(G) = 1/3$$

$$MSNE = \sigma = \left\{ \left(\sigma_1(H) = \frac{2}{3}, \sigma_1(G) = \frac{1}{3} \right); \left(\sigma_2(H) = \frac{2}{3}, \sigma_2(G) = \frac{1}{3} \right) \right\}$$

3. *Find a symmetric correlated equilibrium (probabilities remain the same when we interchange rows for columns) which Pareto dominates the symmetric Nash equilibrium. The correlated equilibrium may use public randomization if you wish, but you must show it is a correlated equilibrium by showing that neither player wishes to deviate from the recommendation of the randomization device.*

   To find a correlated equilibrium that is a Pareto improvement, we look at the payoff profile of the MSNE:
$$0\frac{2}{3}\frac{2}{3} + 2\frac{2}{3}\frac{1}{3} + 1\frac{1}{3}\frac{2}{3} + 0\frac{1}{3}\frac{1}{3} = \frac{2}{3}$$

   The payoff profile is $(2/3, 2/3)$, and so we have to find a correlated equilibria that yields better payoffs for at least one player.



   Many of you rightly pointed out that the payoff matrix is the same as in the previous exercise, so the solution from exercise 3 is applicable here. It would correspond to the payoff profile of $(1, 1)$ in the graph above. There are, as the graph shows, many other correlated equilibria. Let's induce the centralized Pareto efficient outcome: $(3/2, 3/2)$. Recall that, to define a correlated equilibrium, we must specify:

   1. The probability space: $\Omega = \{HG, GH\}$, with $\pi(HG) = \pi(GH) = 1/2$. Notice that, when written in matrix form, these probabilities yield a symmetric equilibrium.

2. The partition over outcomes:

$$P_1 = \{\{HG\}, \{GH\}\}$$
$$P_2 = \{\{HG\}, \{GH\}\}$$

3. The strategies that will induce this equilibrium:

$$\sigma_1(HG) = H \qquad \sigma_1(GH) = G$$
$$\sigma_2(HG) = G \qquad \sigma_2(GH) = H$$

We verify that this correlated equilibrium has no profitable deviations (ie, we check the BR of each player).

$$BR_1(HG) = H \qquad BR_1(GH) = G$$
$$BR_2(HG) = G \qquad BR_2(GH) = H$$

Exercise 5: Trembling Hand Perfection

A strategy profile σ is trembling hand perfect (THP) if there exists a sequence of strategy profiles $\sigma^n \to \sigma$ with $\sigma_i^n(s_i) > 0$ for all $s_i \in S_i$ and all $i \in I$ such that $\sigma_i(s_i) > 0$ implies that s_i is a BR to σ_{-i}^n. Prove that every THP profile is a Nash equilibrium.

WTS: σ is THP $\to$ σ is NE. In other words, THP $\subseteq$ NE.

Direct proof. By definition, σ is THP, so there exists a sequence that satisfy the conditions of the definition. So, for any player i and strategy $s_i \in S_i$ played with positive probability $\sigma_i(s_i) > 0$:

$$u_i\left(s_i, \sigma_{-i}^n\right) \geq u_i\left(s_i', \sigma_{-i}^n\right) \qquad \forall s_i' \in S_i$$

This holds for a trembling sequence. By continuity of $u(\cdot)$, taking the limits:

$$\left.\begin{array}{l} \lim_{\sigma_{-i}^n \to \sigma_{-i}} u_i\left(s_i, \sigma_{-i}^n\right) = u_i(s_i, \sigma_{-i}) \\ \lim_{\sigma_{-i}^n \to \sigma_{-i}} u_i\left(s_i', \sigma_{-i}^n\right) = u_i(s_i', \sigma_{-i}) \end{array}\right\} u_i\left(s_i, \sigma_{-i}\right) \geq u_i\left(s_i', \sigma_{-i}\right) \qquad \forall i, s_i' \in S_i$$

Take another strategy s_i'' such that $\sigma_i\left(s_i''\right) > 0$ in the NE. We will also have:

$$u_i\left(s_i'', \sigma_{-i}\right) \geq u_i\left(s_i', \sigma_{-i}\right) \qquad , s_i' \in S_i \tag{2}$$

As s_i is also a NE, we will have that both:

$$\left.\begin{array}{l} u_i\left(s_i'', \sigma_{-i}\right) \geq u_i\left(s_i, \sigma_{-i}\right) \\ u_i\left(s_i, \sigma_{-i}\right) \geq u_i\left(s_i'', \sigma_{-i}\right) \end{array}\right\} u_i\left(s_i, \sigma_{-i}\right) = u_i\left(s_i'', \sigma_{-i}\right) \tag{3}$$

The two conditions 2 and 3 imply that σ is a NE. [6] □

[6] These two conditions are the ones stated in the definition of NE in the supplementary notes.

Give an example of a Nash equilibrium in a 2x2 game which is not trembling hand perfect and explain why.

The key here is that a NE that contains a weakly dominated strategy is not THP. Consider the following game:

		P2	
P1		L	R
	U	1,1	0,0
	D	1,0	2,1

$$PSNE = \{(U, L), (D, R)\}$$

However, (U, L) is not THP as U is weakly dominated by D. To see this, take any fully mixed strategy σ_2 for P2 and evaluate P1's payoffs:

$$u_1(U, \sigma_2) = \sigma_2(L) < \sigma_2(L) + 2\sigma_2(R) = u_1(D, \sigma_2) \forall \sigma_2(R) > 0$$

Thus, there are no fully mixed strategy sequences that yield U as a BR for P1. Any positive probability for P2 to play R makes P1 strictly prefer D.

Exercise 6: Becker

There are 2 groups, each making a non-negative bid b_k. The utility of group k is:

$$u_k = (b_k - b_{-k}) - \beta\frac{(b_k - b_{-k})^2}{2} - \frac{c_k b_k^2}{2}$$

a.&b.&c. *Show that a Nash equilibrium exist and it is unique. When is it interior?*

Recall the proposition from the supplementary notes:

Proposition. A NE exists [7] in game $\Gamma = [I, \Delta S_i, u_i(\cdot)]$ if for all $i = 1, \ldots, I$:

- S_i is a nonempty convex, and compact subset of some Euclidean space R^M.
- $u_i(s_1, \ldots, s_I)$ is continuous in $S = (s_1, ..., s_I)$ and quasi-concave in s_i

The strategies in this case are bids.

- Bids must be compact: closed and bounded. In principle, $b_k \in [0, \infty]$. By economic theory, no bidder can bid an infinite amount of resources. So bids are bounded by an arbitrarily large enough B: $b_k \in [0, B]$. In addition, b_k is trivially convex and non empty.
- u_i is trivially continuous in S. For quasi-concavity, note:

$$\frac{\partial^2 u_k}{\partial b_k^2} = -\beta - c_k < 0$$

u_k is concave and hence quasi-concave.

[7] Alternatively, you can simply find the Nash equilibrium and this will show it exists.

We have now shown that the NE **exists**. We now move to show that it is interior and it is unique. To do this, we find the NE.

Note that $b = (b_k, b_{-k})$ is a NE iff $b_k \in BR_k(b_{-k})$. For a given b_{-k} b_k is the best response correspondence if $u_k(b_k, b_{-k}) \geq u_k(b'_k, b_{-k})$ for all $b'_k \in [0, B]$. This is exactly the $\max_{b_k} u_k$ given b_{-k}. The conditions we imposed on the existence of the NE guarantee that we can find such max.

$$\frac{\partial u_k}{\partial b_k} : 1 - \beta(b_k - b_{-k}) - c_k b_k = 0$$
$$BR_k(b_{-k}) = b_k = \frac{1 + \beta b_{-k}}{c_k + \beta}$$
$$BR_{-k}(b_k) = b_{-k} = \frac{1 + \beta b_k}{c_{-k} + \beta}$$

These functions are both single-valued and linear. This means that, for positive values of c_k, c_{-k}, they must intercept only once. [8] Hence, the NE is unique.

To study whether it is interior, we look at the solution to the fixed point.

$$b_k^\star = BR_k(BR_{-k}(b_k^\star))$$
$$b_k^\star = \frac{c_{-k} + 2\beta}{c_k c_{-k} + \beta(c_k + c_{-k})}$$

Given the parameter values (strictly positive by assumption), bidding zero is never a BR.

$$\forall c_k, c_{-k}, \beta > 0 \qquad b_k^\star \neq 0 \quad \text{as } BR_k(0) > 0$$

Hence, the NE is interior.

e. *Higher costs lead to lower bids.*

$$\frac{\partial b_k^\star}{\partial c_k} = -\frac{(c_{-k} + 2\beta)(c_{-k} + \beta)}{(c_k c_{-k} + \beta(c_k + c_{-k}))^2} < 0 \qquad \forall (c_k, c_{-k}, \beta) \in \mathbb{R}^3_{++}$$
$$\frac{\partial b_k^\star}{\partial c_{-k}} = -\frac{\beta c_k + 2\beta^2}{(c_k c_{-k} + \beta(c_k + c_{-k}))^2} < 0 \qquad \forall (c_k, c_{-k}, \beta) \in \mathbb{R}^3_{++}$$

Thus, with positive cost parameters, both bids are decreasing in both costs. The statement is true.

f. *Less efficiency leads to lower transfers.*
Taking β as the **in**efficiency parameter, and interpreting as transfer the difference between the two bids:

$$T = b_k^\star - b_{-k}^\star = \frac{c_{-k} - c_k}{c_k c_{-k} + \beta(c_k + c_{-k})}$$
$$\frac{\partial T}{\partial \beta} = -\frac{c_{-k}^2 - c_k^2}{(c_k c_{-k} + \beta(c_k + c_{-k}))^2}$$

[8] To see this, you can express both BRs in terms of one of the bids:

$$BR_k(b_{-k}) = b_k = \frac{1 + \beta b_{-k}}{c_k + \beta} \qquad \longrightarrow \qquad b_{-k} = \frac{c_k + \beta}{\beta} b_k - \frac{1}{\beta}$$
$$BR_{-k}(b_k) = b_{-k} = \frac{1 + \beta b_k}{c_{-k} + \beta} \qquad \longrightarrow \qquad BR_{-k}(b_k) = \frac{\beta}{c_{-k} + \beta} b_k + \frac{1}{\beta}$$

The slope of one is the inverse of the other. For strictly positive parameters, the functions intercept only once.

The sign of $\partial T/\partial \beta$ depends on the relationship between c_k and c_{-k}.

- When $c_k > c_{-k}$, then $b^\star_{-k} > b^\star_k$ so that $T < 0$ and $\partial T/\partial \beta > 0$. When transfers are negative, an increase in β increases the transfer.
- When $c_k < c_{-k}$, then $b^\star_{-k} < b^\star_k$ so that $T > 0$ and $\partial T/\partial \beta < 0$. When transfers are positive, an increase in β decreases the transfer.

This means that less efficiency (a higher β) decreases the absolute transfer when the transfer is positive, and increases the absolute transfer when the transfer is negative: less efficiency leads to lower transfers in absolute terms, and the statement is correct.

$$\frac{\partial |b^\star_k - b^\star_{-k}|}{\partial \beta} < 0$$