The Reputation Trap[1]

David K. Levine[2]

Abstract

Few want to do business with a partner who has a bad reputation. Consequently once a bad reputation is established it can be difficult to get rid of. This leads on the one hand to the intuitive idea that a good reputation is easy to lose and hard to gain. On the other hand it can lead to a strong form of history dependence in which a single beneficial or adverse event can cast a shadow over a very long period of time. It gives rise to a reputational trap where an agent rationally chooses not to invest in a good reputation because the chances others will find out is too low. Never-the-less the same agent with a good reputation will make every effort to maintain it. Here a simple reputational model is constructed that has a unique equilibrium and the conditions for a reputation trap are characterized.

[1]I am indebted to Drew Fudenberg, Juan Block, Rohan Dutta, Andrea Mattozzi, and Salvatore Modica. Financial support from the EUI Research Council is gratefully acknowledged.
[2]Department of Economics and RSCAS at the EUI, WUSTL; email: david@dklevine.com

1. Introduction

It is conventional to think that a good reputation is easy to lose and hard to gain. One reason we suspect this might be the case is that with a good reputation people will be eager to do business so if they are cheated it will quickly become known. On the other hand nobody wants to do business with a bad reputation so even if honest behavior takes place nobody will find out. In such a setting it is intuitive that history matters. If an adverse event causes a loss of reputation the difficulty of restoring it provides little incentive for honesty, so the bad reputation will deservedly remain so long after the circumstances that caused it are gone. On the other hand, there are reasons for honesty besides reputation - if circumstances dictate honesty it will take a long time before others find out, but once they do reputation will be restored - and even after the circumstances dictating honesty are gone it will be desirable to continue to be honest to avoid losing reputation. In other words, once reputation is restored it will also persist. Consequently, two otherwise identical individuals may find themselves with entirely different incentives for honesty because of an adverse or beneficial event that happened in the distant past. There is, as we shall see, a rather important hole in this intuition.

The goal of this paper is to develop a model that captures the intuitive idea that differential observability leads to history dependence. In doing so we draw key elements from the reputational literature. Following the gang-of-four[3] we introduce behavioral types - as in [Mailath and Samuelson 2001] these type are persistent but not completely so. We allow for good types (beneficial events) as in the gang-of-four and bad types (adverse events) as in [Mailath and Samuelson 2001]. Finally, following an idea in [Fudenberg and Levine 1989] we assume that the short-run players face an entry decision and that the information generated about long-run player behavior is greater if the short-run player chooses to enter than if not. This observational asymmetry leads to an important change from the [Mailath and Samuelson 2001] model where good and bad events are symmetric and reputation is equally easily lost or restored.

This model captures the intuitive elements of persistent reputation if we add two additional assumptions concerning the short-run player. As is standard in these types of models in each period a single representative short-run player is a stand-in for a large population of players. This implies myopic behavior. Here we also take it to mean lack of ability of the short-run players to coordinate. On the one hand lack of coordination means that the representative short-run player cannot mix and is limited to pure strategies. On the other hand response to events in the distant past also cannot be coordinated. We model this as limited memory. Without these assumptions, as indicated, there are holes in the basic intuition of the first paragraph. With unlimited memory if short-run players stay out and no information is generated it eventually becomes likely that the long-run player has migrated back to a "normal"type. It is now possible for the

[3] [Kreps and Wilson 1982] and [Milgrom and Roberts 1982]

short-run players and long-run player to coordinate. On a particular date it is common knowledge that if the long-run player is normal honest behavior will take place and that the short-run player will enter. This is then a self-fulfilling prophecy.[4] It is not, however, a very compelling one: it requires that both players agree about the exact timing of events in the long-distant past and that they agree that "today is the day." This we rule out through the assumption of limited memory. Second, with limited memory mixing can create a signal jamming problem akin to that in the cheap talk literature[5] This is ruled out in our setting by the assumption that the short-run player cannot mix.

Reputation theory has wide application to a variety of settings, but an important motivation for this research is a puzzle in the political economy of culture and institutions concerning the persistence of dysfunctional cultures. On the one hand there is a substantial literature indicating that these can be quite persistent. [Acemoglu and Robinson 2001] give evidence for persistence on the order of four centuries. [Bigoni et al 2013] have evidence of a similar effect over nearly nine centuries. [Dell and Querubin 2018] have highly persuasive evidence for persistence on the order of a century and a half.

On the other hand it cannot be that it is simply impractical to change social and cultural norms: side by side with the survival of dysfunctional norms we see abrupt change over periods of a few decades. Two central aspects of culture are religion and language. Yet we observe that even these fundamental aspects of society change over short periods of time. Prior to 1990 the country of Ireland could as well be described as Catholic as Irish. Yet by the end of the decade the church lost its central place in Irish life and the country could be better described as secular.[6] With respect to language we may point to the remarkable example of Hebrew. In 1880 Hebrew was not a conversational language. In 1903 there were perhaps a few hundred Hebrew speakers. Within fifteen years more than 30,000 Jews in Palestine claimed Hebrew as their native language.[7]

While religion and language are important elements of culture their role in economic life is controversial. Are similar abrupt changes in economic culture possible? Before asserting that it is absurd to imagine that Nigerians could enrich themselves by replacing their own culture with Japanese culture, we might ask how Japanese culture came to be what it is. In 1868 Japan had a culture, technology, and standard of living similar to medieval Europe. By 1904 when Japan shocked the world by defeating a major Western military power it had transformed itself into a modern industrial state. The stunning transformation to a Western culture during the roughly 40 years of the Meiji era affected virtually every aspect of life in Japan.[8]

The reputational theory of this paper offers a possible reconciliation between the ideas that dysfunctional norms may persist over a long period of time and

[4] In [Acemoglu and Wolitzky 2012] this induces a cycle.
[5] See, for example, [Crawford and Sobel 1982].
[6] See, for example, [Donelly and Inglish 2010].
[7] See, for example, [Bar-Adon 1972].
[8] See, for example, [Jansen 2002].

the fact it is not impractical to change them. Insofar as the importance of good social norms for economic success revolves around good treatment of immigrants and foreign investors there should indeed be a reputational effect. A region that has a reputation for poor treatment of foreigners is unlikely to get much immigration or foreign investment and so is unlikely to have thriving urban centers of production and innovation. It then becomes the case that even if treatment of outsiders is improved nobody is likely to find out, so indeed the dysfunctional norm of cheating outsiders may be a form of reputation trap. I do not mean to argue that a reputation trap is the only reason dysfunctional cultures do not change: there are many costs associated with changing an entire culture and in many circumstances it may not be worth it. Never-the-less given the enormous disparity in income between Nigeria and Japan we may well ask if the cost of change is the entire reason for its absence or whether a reputation trap may not play a role as well.

Much recent work on reputation has also focused on finding equilibria rather than computing bounds. However, unlike here mixed strategies have played a key role. [Mathevet, Pearce and Stachetti 2019] examine an information design problem where the behavioral type mixes. In [Phelan 2006] it is the normal type that mixes. This leads to trust that is only gradually regained, but it does not lead to a reputation trap.

In a sense the phenomena our theory endeavors to explain is connected to the literature on poverty traps. That literature[9] is based on a different mechanism: it is based on the idea that there are increasing returns to scale in the accumulation of human capital. The remedies for this type of poverty trap are quite different than for a reputation trap. We do not have a great deal of evidence about the relative importance of human and social capital, but we do have the estimates of [Dell and Querubin 2018] that only about one third of the persistence of poverty is due to human capital so there is substantial scope for a reputational mechanism.

The reputational ideas here are also related to the literature on self-confirming equilibrium.[10] In that literature also a trap can arise because of the difficulty of drawing causal inferences about off-the-equilibrium path play of an opponent. Here we incorporate that idea into a model of rational Bayesian learning with imperfect observability and the uniqueness of equilibrium in our model enables us to draw sharp results.

2. The Model

A dynamic game is played between overlapping generations of finitely lived players. There are two player roles: player 1 is a long-run player who lives many periods and player 2 represents a mass of short-run players who live a single period. Each period a stage game is played. The long-run player must

[9]See, for example, [Azariadis and Drazen 1999].

[10]See, for example, [Fudenberg and Levine 1993] and [Sargent, Williams and Zha 2006].

first choose whether or not to make a costly investment. Let $a_1 \in \{0,1\}$ denote the decision of the long-run player with 1 meaning to invest and the cost being ca_1 where $0 < c < 1$. The short-run player moves second and without observing the investment choice of the long-run player decides whether to enter $a_2 = 1$ or stay out $a_2 = 0$. The short-run player receives utility 0 for staying out, utility -1 for entering when no investment has been made and utility $V > 0$ for entering when the investment has been made. There are three types $\tau \in \{b, n, g\}$ of long-run player where g means "good" (a beneficial event), b means "bad" (an adverse event), and n means "normal." Player type is fixed during the lifetime of the player. The good and bad types are behavioral types: the good type always invests and the bad type never invests. The stage game payoff of the normal type is given by $a_2 - ca_1$. Players care only about expected average utility during their lifetime.

The life of a long-run player is stochastic: with probability δ the player continues for another period, and with probability $1-\delta$ is replaced. When a long-run player is replaced the type may change. The probability type τ is replace by a type $\sigma \neq \tau$ is $Q_{\tau\sigma}\epsilon/(1-\delta) > 0$. We are interested in the case in which types are persistent - that is, in which ϵ is small.

At the beginning of each period a public signal z of what occurred in the previous period is observed and takes on one of three values: $1, 0, N$. If entry took place last period the signal is equal to last-period long-run player investment. If the short-run player stayed out last period then with probability $1 \geq \pi > 0$ the signal is equal to last period long-run player investment and with probability $1 - \pi$ the signal is N. In other words, when the short-run player stays out less information is generated about the behavior of the long-run player.

Players are only aware of events that occur during their lifetime. Moreover, we assume that the short-run players being a short-lived mass are unable to coordinate on a mixed strategy. Hence a strategy for a short-run player is a deterministic choice of action $\alpha_2(z) \in \{0,1\}$ as a function of the beginning of period signal. The long-run player observes finite histories h of lifetime events and chooses a probability of investment $\alpha_1(h) \in [0,1]$.

Since players know only events that have taken place during their lifetime they do not know how long the game has been going on. They must base their decision only on what they have observed - not on time or their own identity. We take this to mean that there is one common strategy used by all short-run players and one common strategy used by all long-run players. Note that this assumption is different than Markov perfection because not all the information on which players base their decisions is payoff relevant. In particular if two sufficiently patient normal type long-run players played against each other in a prisoner's dilemma stage game in which last period play is observed one equilibrium would be the grim-strategy equilibrium.

Throughout the paper we will assume *generic cost* in the sense that

$$c \notin \left\{ \delta, \frac{\delta}{2-\pi}, \frac{\delta\pi}{1-\delta+\delta\pi}, \frac{\delta\pi(\pi-\delta\pi)}{(1-\delta\pi)(1-\delta)) + \delta\pi(\pi-\delta\pi)} \right\}.$$

Beliefs and Equilibrium

Given a strategy $\alpha_2(z)$ of the short-run player the long-run player faces a Markov decision problem. In each period given the signal z a short-run player action $\alpha_2(z)$ will result. The optimization problem faced by the long-run player depends only on that action. Let $V(a_2)$ denote the corresponding expected average value of utility. First period utility is $a_2 - ca_1$. With probability δ the game continues and the probability of the next signal is $P(z'|z, a_1)$ where $P(1|z, 1) = P(0|z, 0) = \alpha_2(z)+(1-\alpha_2(z))\pi$ and $P(N|z, a_1) = (1-\alpha_2(z))(1-\pi)$. Hence the Bellman equation is

$$V(a_2) = \max_{a_1}(1-\delta)\left[a_2 - ca_1\right] + \delta \sum_{z'} P(z'|z, a_1) V(\alpha_2(z')).$$

In Lemma 1 of the Appendix we show that for small ϵ this problem has a unique solution - the long-run player has a strict best response to the short-run player. If a Markov decision problem has a unique solution it is necessarily a pure Markov strategy: that, is we may assume that the long-run player chooses $\alpha_1(z) \in \{0, 1\}$ depending only on the signal. This fact considerably simplifies the analysis of short-run player optimality.

Given that both the long-run and short-run players are following pure Markov strategies the joint distribution of types and signals $\mu_{z\tau}(t)$ follows a Markov process. If the short-run player chooses to remain out $\alpha_2(z) = 0$ for all z then the signal N cannot occur; with this exception every state is reachable with positive probability from every other state so the Markov process is ergodic on the relevant state space with $\mu_{z\tau}(t)$ converging to a unique limit $\mu_{z\tau}$. As the short-run player has no knowledge of time this unique ergodic limit is taken to be the beliefs of the short-run player prior to observing the signal.

An *equilibrium* consists of strategies $\alpha_1(z), \alpha_2(z)$ together with the unique ergodic beliefs $\mu_{z\tau}$ generated by those strategies such that the long-run player is playing a best response to the short-run player strategy and the short-run player is playing a best response to the ergodic beliefs and the signal.[11]

3. Equilibrium

Theorem 1. *For given V, Q there exists an $\underline{\epsilon} > 0$ such that for $\underline{\epsilon} \min\{\pi, 1-\pi\} > \epsilon > 0$ there is a unique equilibrium, it is strict, and the short-run player enters only on the good signal. There are three mutually exclusive types of equilibria*

a. if $\delta < c$ the long-run player never invests, otherwise

b. if $\pi < (c-\delta c)/(\delta - \delta c)$ the long-run player invests only on the good signal

c. if $\pi > (c-\delta c)/(\delta - \delta c)$ the long-run player always invests.

Note that the boundary cases are ruled out by the generic cost assumption.

[11] We can give a general definition of equilibrium when there is not a unique ergodic limit by defining the set of beliefs that are a limit point of the signal/type process for some initial condition, then requiring that the short-run player strategy be optimal with respect to one of these.

The proof can be found in the Appendix. The equilibrium itself is relatively intuitive. The assumption that ϵ is small means that types are highly persistent so the short-run player does not put much weight on the possibility of the type changing. Given the possible strategies of the long-run player the signal 0 indicates either a bad type or a normal type who will not invest if entry is not anticipated. Hence it makes sense for the short-run player not to enter in the face of bad signal. Similarly the signal 1 indicates either a good type or a normal type who will invest if entry is anticipated, so it makes sense for the short-run player to enter in the face of a good signal.

More subtle is the inference of the short-run player when the signal N is observed. The short-run player can infer that the previous short-run player chose not to enter - hence must have received the bad signal or was in the same boat with the signal N. As a result while less decisive than the signal 0 the signal N also indicates past bad behavior by the long-run player, so staying out is a good idea.

For the long-run player the choice is whether to invest when entry is anticipated and when it is not. Consider a modified problem for a long run player deciding whether to invest: a cost c can be incurred resulting in probability p of successfully establishing a good reputation and gaining $1-c$ in the future. Here the expected average present value of the gain from investment is $\Gamma = -(1-\delta)c + \delta p(1-c) + \delta(1-p)\Gamma$ or

$$\Gamma = \frac{\delta p(1-c) - (1-\delta)c}{1-\delta(1-p)}.$$

If this is negative, that is $\delta p(1-c) < (1-\delta)c$ then it is best not to invest and conversely. Take first the case where information is revealed immediately, that is $p = 1$. In this case the condition for not wishing to invest is $\delta < c$. This is a standard case, corresponding to part (a) of the Theorem in which the long-run player is impatient and does not find it worthwhile to give up c for a future gain of $1-c$. In this case investment will only take place only occasionally during beneficial events when the good type finds it optimal to invest for non-reputational reasons.

When $\delta > c$ it is worth it to maintain a reputation when the short-run player enters as indeed in this case $p = 1$. The remaining question is whether it is also worth it to invest when the short-run player does not enter. In this case $p = \pi$, and the condition for investment is that given in (b) and (c). If π is high enough so the positive news spreads quickly then it is worth investing even when the short-run player does not enter. This corresponds to the "usual" reputational case, for example in [Kreps and Wilson 1982], [Milgrom and Roberts 1982], [Fudenberg and Levine 1989], [Fudenberg and Levine 1992] or [Mailath and Samuelson 2001]. Here the long-run player always is willing to invest. Occasionally an adverse event occurs and the bad type finds it unprofitable to invest so investment does not take place until another normal or good type arrives.

The new case and the interesting case is case (b) where $\delta > c$ so the interest rate is high enough to maintain a reputation, but $\pi < (c - \delta c)/(\delta - \delta c)$ so it is not worth it to try to acquire a reputation. Here we have strong history dependence. Depending on the history a normal type will be in one of two very different situations. A normal type that follows a history of good signals, will invest, have a good reputation and have a wealthy and satisfactory life with an income of $1 - c$. A normal type that has the ill-luck to follow a history in which the last signal was bad will not invest, will have a (deservedly) bad reputation, and have an impoverished life with an income of 0. This we may think of as a reputational trap. The only different between these normal types is an event that took place in the far distant past: did the last behavioral type correspond to an adverse or beneficial event? Looked at another way, adverse and beneficial events, rare as they are, cast a very long shadow. After a beneficial event there will be many lives of prosperous normal types - indeed until an adverse event occurs. Contrariwise, following an adverse event normal types will be mired in the reputation trap until they are fortunate enough to have a beneficial event. Hence, for example, an outside threat that causes people to pull together (a beneficial event) may have very long-term consequences indeed.

The Role of Behavioral Types

To better understand the role of behavioral types, consider their absence. As usual the static Nash equilibrium - always stay out and never invest - is an equilibrium. In case (a) of Theorem 1 this is the only equilibrium. For higher discount factors both the case (b) and case (c) strategies are Nash equilibria, although the only one that is subgame perfect is the case (b) equilibrium in case (b). In the usual way the presence of good types eliminates the static Nash equilibrium once the discount factor is high enough. The bad types, however, are key in selecting between the (b) and (c) type equilibria. The presence of behavioral types insures that the ergodic distribution is unique and that all signals (except possibly N) are present - so acts somewhat like trembles. The non-subgame perfect type (c) equilibrium is eliminated in case (b) and type (b) equilibrium in case (c) because play must be optimal following a signal of no investment. Most striking is the type (c) equilibrium in case (c). Despite the fact that the normal types always invest it is optimal for the short-run player to stay out on a signal of no investment: this is because such a signal indicates a bad type.

4. Mixed Strategies

The analysis here is *de facto* limited to pure strategy equilibria. We imposed that limitation by restricting the short-run player; we did so because it makes sense that a mass of short-run players would have difficulty in achieving the coordination needed to mix. However, it is the lack of mixing by the long-run player that is critical to the analysis - and restricting the short-run player to pure strategies makes it sub-optimal for the long-run player to mix.

To understand this better, consider first that if there are few normal types of long-run players we can prove that there is no mixed equilibrium. Specifically, take $\Pr(\tau|n)$ to be fixed rather than depending on ϵ so as $\epsilon \to 0$ the ergodic marginal $\mu_n \to 0$. In the limit where there are only behavioral types the Appendix shows that it is a strict best response for the short-run player to enter on good and stay out on bad or no signal. Hence this must remain true for ϵ near zero, in which case μ_n is small but not zero. However, as indicated, once we know that the short-run player can only play a pure strategy, the same is true of the long-run player and equilibrium is unique. Unfortunately the case of few normal types is the least interesting case.

What then goes wrong as we increase the number of normal long-run types? Why can this lead to the possibility of a mixed equilibrium? The problem is a signal jamming problem. If the normal long-run type mixes and there are few behavioral types then the signal from last period - whatever it may be - is far more likely to come from a normal type who is mixing than a behavior type. Hence reputational arguments based as they are on inferences about behavioral types break down.

This analysis, however, shows that the possibility of signal jamming is an artifact of a single period of memory. That is, with longer memory, it is easier to distinguish a behavioral type from a normal type who mixes: this is the basis of the payoff bound with unlimited persistence and high patience derived in [Fudenberg and Levine 1992] and in the limited persistence case by [Ekmekci, Gossner and Wilson 2012]. These results apply to the limiting case of this model and imply that when there is enough patience (so in particular the normal long-run player always invests) the normal long-run player does not get much less than $1-c$ in average present value in any equilibrium.

Turning away from the details of what a mixed strategy equilibrium might be and when it might exist, would such an equilibrium make sense? As we have indicated, actual mixing by the short-run players probably does not. However, fractional entry as in [Harsanyi 1973] would and in the simplest case this would be payoff and signal equivalent to mixing. Against this, are the learning results from [Block, Fudenberg and Levine 2018] indicating that when mixed and strict equilibria co-exist the strict equilibria are likely to be far more commonly observed.

5. Conclusion

If the reputation trap is real we should ask the public policy question of how to get out of it. For example, if Southern Italy is caught in a reputation trap, what might the central government of Italy or the EU do to help? One possibility is to subsidize the cost of investment: if the cost c is low enough then investment even with the bad signal will be profitable and - eventually - the trap will be escaped. Welfare analysis of the model, however, indicates that this is probably not a good idea. The long-run player already has the possibility of making the investment and finds it not worth while; if the money designated for an investment subsidy was instead given to the long-run player the long-run player would choose not to spend it on investment - and would be strictly better off.

The model, however, points to another possible direction: if π could be increased it would be much easier to escape the reputation trap. Here an outside agency might have an advantage over the long-run agent having, perhaps, greater influence on outsiders and information flow to outsiders. Large events such as a World Cup or the Olympics come to mind in this context. By bringing large numbers of outsiders a cultural change is publicized - and that possibility increases the incentive for the change. One reason cities and regions compete for these events is precisely in hopes of obtaining favorable publicity. We need to ask, however, if this has ever worked as a means of escaping a reputation trap. Certainly to be effective the investment must actually take place - hence the Olympics in Athens in 2004 or in Rio in 2016 simply confirmed what everybody already believed about those cities. In this context it must also be emphasized that to be effective the increase in π must be large enough - it must cross the threshold for which it becomes profitable to invest on the bad signal. Possible positive examples are the Olympics in Barcelona in 1992 and the World Exposition in Chicago in 1893: however, at the moment no satisfactory empirical analysis of these events exists.

References

[1] Acemoglu, D., and Wolitzky, A. (2012). Cycles of Distrust: An Economic Model (No. w18257). NBER.

[2] Acemoglu, D., Johnson, S., and Robinson, J. A. (2001). The colonial origins of comparative development: An empirical investigation. *American economic review*, 91(5), 1369-1401

[3] Azariadis, C., and Drazen, A. (1990). Threshold externalities in economic development. *The Quarterly Journal of Economics*, 105(2), 501-526.

[4] Bar-Adon, A. (1972). Sy Agnon and the Revival of Modern Hebrew. *Texas Studies in Literature and Language*, 14(1), 147-175.

[5] Bigoni, M., Bortolotti, S., Casari, M., Gambetta, D., and Pancotto, F. (2013). Cooperation hidden frontiers: The behavioral foundations of the Italian North-South divide, mimeo Bologna.

[6] Block, J. I., Fudenberg, D. and Levine, D. K. (2018). Learning dynamics with social comparisons and limited memory. *Theoretical Economics*.

[7] Crawford, V. P., and Sobel, J. (1982). Strategic information transmission. *Econometrica*, 1431-1451.

[8] Dell, M., Lane, N., and Querubin, P. (2018). The historical state, local collective action, and economic development in Vietnam. *Econometrica*, 86(6), 2083-2121.

[9] Ekmekci, M., Gossner, O., and Wilson, A. (2012). Impermanent types and permanent reputations. *Journal of Economic Theory*, 147(1), 162-178.

[10] Donnelly, S., and Inglis, T. (2010). The media and the Catholic Church in Ireland: Reporting clerical child sex abuse. *Journal of Contemporary Religion*, 25(1), 1-19.

[11] Fudenberg, D., and Levine, D. K. (1989). Reputation and Equilibrium Selection in Games with a Single Long-Run Player. *Econometrica*, 57.

[12] Fudenberg, D., and Levine, D. K.. (1992). Maintaining a Reputation when Strategies are Imperfectly. *Review of Economic Studies*, 59, 561-579.

[13] Fudenberg, D., and Levine, D. K. (1993). Self-confirming equilibrium. *Econometrica*, 523-545.

[14] Harsanyi, J. C. (1973). Games with randomly disturbed payoffs: A new rationale for mixed-strategy equilibrium points. *International journal of game theory*, 2(1), 1-23.

[15] Jansen, M. B. (2002). *The making of modern Japan*. Harvard University Press.

[16] Kreps, D. M., and Wilson, R. (1982). Reputation and imperfect information. *Journal of economic theory*, 27(2), 253-279.

[17] Mailath, G. J., and Samuelson, L. (2001). Who wants a good reputation? *The Review of Economic Studies*, 68(2), 415-441.

[18] Mathevet, L., Pearce, D. and Stachetti, E. (2019). Reputation and Information Design, NYU

[19] Milgrom, P., and Roberts, J. (1982). Predation, reputation, and entry deterrence. *Journal of economic theory*, 27(2), 280-312.

[20] Phelan, C. (2006). Public trust and government betrayal. *Journal of Economic Theory*, 130(1), 27-43.

[21] Sargent, T., Williams, N., and Zha, T. (2006). Shocks and government beliefs: The rise and fall of American inflation. *American Economic Review*, 96(4), 1193-1224.

Appendix: Proof of the Main Theorem

As the analysis of the long-run player is quite standard, we state the main result with the proof in the online Appendix.

Lemma 1. *The optimum for the long run-player is strict so pure and depends on the state only through* a_2. *The strategy* $\alpha_1(0) > \alpha_1(1)$, *is strictly sub-optimal. If* $\alpha_2(0) \geq \alpha_2(1)$ *it is strictly optimal for the long-run player never to invest, that is* $\alpha_1(z) = 0$.

Next we examine the beliefs of the short-run player. For given pure strategies of both players the signal type pairs (z, τ) are a Markov chain with transition probabilities independent of δ and depending only on ϵ, π and the strategies of the two players. Excluding the state N in case the short-run player always enters the chain is irreducible and aperiodic so it has a unique ergodic distribution $\mu_{z\tau}$. We first analyze the marginals $\mu_\tau$and μ_z.

Lemma 2. *The marginals* μ_τ *are independent of* ϵ. *Let* $\underline{\mu} = \min_\tau \mu_\tau$. *Then* $\underline{\mu} > 0$, $\mu_0, \mu_1 \geq \pi\underline{\mu}$, *if* $a_2(0) = a_2(1) = 1$ *then* $\mu_N = 0$, *otherwise* $\mu_N \geq (1-\pi)\underline{\mu}$.

Proof. The type transitions are independent of the signals, so we analyze those first. For $\epsilon > 0$ we have $\mu_\tau > 0$ since every type transition has positive probability. This ergodic distribution is the unique fixed point of the 3×3 transition matrix A, which is to say given by the intersection of the null space of $I - A$ with the unit simplex. Since $A = I + Q\epsilon$ it follows that it is given by the intersection of the null space of $Q\epsilon$ with the unit simplex. As the null space of $Q\epsilon$ is independent of ϵ the marginals μ_τ are independent of ϵ as well.

For the signals we have $\mu_1 \geq \pi\mu_g$ and $\mu_0 \geq \pi\mu_b$. If if $a_2(0) = a_2(1) = 1$ then the state N is transient. If $a_2(1) = 0$ then $\mu_N \geq (1-\pi)\mu_g$ while if $a_2(0) = 0$ then $\mu_N \geq (1-\pi)\mu_b$. ☐

It will be convenient to normalize so that $\max(\mu_\sigma/\mu_\tau)Q_{\tau\sigma} = 1$. Next we find the conditional probabilities $\mu_{z|\tau}$.

Lemma 3. *When* $z = N$

$$\mu_{N|\tau} = (1-\pi)\left(\sum_y 1(a_2(y) = 0)\mu_{y|\tau} + \epsilon H_{N\tau}\right)$$

when $z \neq N$

$$\mu_{z|\tau} = \sum_y 1(a_1(\tau, y) = z)\left[1(a_2(y) = 1) + \pi 1(a_2(y) = 0)\right]\mu_{y|\tau} + \epsilon H_{z\tau}.$$

where $|H_{z\tau}| \leq 2$.

Proof. For given strategies of the players define $P(z, \sigma|y, \tau)$ to be the conditional probability that $z_{t+1} = z, \sigma_{t+1} = \sigma$ conditional on $z_t = y, \tau_t = \tau$. We have

$$\mu_{z\tau} = \mu_{z|\tau}\mu_\tau = \sum_\sigma \sum_y P(z|y,\sigma)P(\tau|\sigma)\mu_{y\sigma}$$

$$= \sum_\sigma \sum_y P(z|y,\sigma) \Pr(\tau|\sigma) \mu_{y|\sigma} \mu_\sigma = \sum_\sigma P(\tau|\sigma)\mu_\sigma \sum_y P(z|y,\sigma)\mu_{y|\sigma}.$$

Since we know that $\mu_\tau > 0$ we may divide to find

$$\mu_{z|\tau} = \sum_\sigma P(\tau|\sigma)\frac{\mu_\sigma}{\mu_\tau} \sum_y P(z|y,\sigma)\mu_{y|\sigma}$$

$$= \sum_\sigma P(\tau|\sigma)\frac{\mu_\sigma}{\mu_\tau} \sum_y P(z|a_2(y), a_1(\sigma, y))\mu_{y|\sigma}.$$

Define $h(\tau|\tau) = -\sum_{\sigma \neq \tau} Q_{\tau\sigma} = (P(\tau|\tau) - 1)/\epsilon$ and for $\tau \neq \sigma$ define $h(\tau|\sigma)\epsilon = (\mu_\sigma/\mu_\tau)Q_{\sigma\tau} = P(\tau|\sigma)/\epsilon$. Observe that h depends only on Q and that

$$|h(\tau|\sigma)| \leq \max\{2(\mu_\sigma/\mu_\tau)Q_{\tau\sigma}|\tau \neq \sigma\} = 2.$$

Then

$$\mu_{z|\tau} = \sum_y P(z|a_2(y), a_1(\sigma,\tau))\mu_{y|\tau} + \epsilon \sum_\sigma h(\tau|\sigma) \sum_y P(z|a_2(y), a_1(\sigma, y))\mu_{y|\sigma}.$$

For $z = N$ this is

$$\mu_{N|\tau} = \sum_y (1-\pi)1(a_2(y) = 0)\mu_{y|\tau} + \epsilon \sum_\sigma h(\tau|\sigma) \sum_y (1-\pi)1(a_2(y)=0)\mu_{y|\sigma}$$

$$= (1-\pi)\left(\sum_y 1(a_2(y)=0)\mu_{y|\tau} + \epsilon H_{N\tau}\right).$$

For $z \neq N$ this is

$$\mu_{z|\tau} = \sum_y P(z|a_2(y), a_1(\sigma,\tau))\mu_{y|\tau} + \epsilon \sum_\sigma h(\tau|\sigma) \sum_y P(z|a_2(y), a_1(\sigma, y))\mu_{y|\sigma}$$

$$\sum_y 1(a_1(\tau, y) = z)\left[1(a_2(y)=1) + \pi 1(a_2(y)=0)\right]\mu_{y|\tau} + \epsilon H_{z\tau}.$$

In both cases $|H_{z\tau}| \leq 2$. □

Lemma 4. *Suppose $a_2(a_1) = 0$ for some $a_1 \in \{0,1\}$. Then*

$$\mu_N \geq \frac{1-\pi}{2}\left(1 - \frac{4\epsilon}{\pi}\right)\underline{\mu}.$$

Proof. Let τ be the type that plays a_1. We have

$$\mu_{a_1|\tau} = \sum_y 1(a_1(\tau, y) = a_1)\left[1(a_2(y)=1) + \pi 1(a_2(y)=0)\right]\mu_{y|\tau} + \epsilon H_{z\tau}$$

$$\mu_{N|\tau} = (1-\pi)\left(\sum_y 1(a_2(y)=0)\mu_{y|\tau} + \epsilon H_{N\tau}\right)$$

For $a_2(N) = 0$ these are

$$\mu_{N|\tau} \geq (1-\pi)\left(\mu_{N|\tau} + \mu_{a_1\tau} + \epsilon H_{N\tau}\right)$$

$$\mu_{a_1|\tau} \geq \pi + \epsilon H_{a_1\tau} \geq \pi - 2\epsilon.$$

which implies

$$\mu_{N|\tau} \geq \frac{1-\pi}{\pi}\left(\mu_{a_1|\tau} + \epsilon H_{N\tau}\right) \geq (1-\pi)\left(1 - \frac{4\epsilon}{\pi}\right).$$

For $a_2(N) = 1$ these are

$$\mu_{N|\tau} \geq (1-\pi)\left(\mu_{a_1|\tau} + \epsilon H_{N\tau}\right)$$

$$\mu_{a_1|\tau} \geq \mu_{N|\tau} + \pi(1-\mu_{N|\tau}) + \epsilon H_{z\tau}$$

which implies

$$\begin{aligned}\mu_{N|\tau} &\geq (1-\pi)\left(\mu_{N|\tau} + \pi(1-\mu_{N|\tau}) + \epsilon H_{z\tau} + \epsilon H_{N\tau}\right)\\ &\geq \frac{1-\pi}{\pi + \pi(1-\pi)}\left(\pi + \epsilon H_{z\tau} + \epsilon H_{N\tau}\right)\\ &\geq \frac{1-\pi}{2-\pi}\left(1 - \frac{4\epsilon}{\pi}\right) \geq \frac{1-\pi}{2}\left(1 - \frac{4\epsilon}{\pi}\right).\end{aligned}$$

The result now follows from $\mu_N \geq \mu_{N|\tau}\mu_\tau \geq \mu_{N|\tau}\underline{\mu}$. □

Lemma 5. *A long-run type τ that plays action a_1 regardless of the signal has*

$$\mu_{\tau|-a_1} \leq \frac{2}{\underline{\mu}}\left(\frac{\epsilon}{\pi}\right)$$

and if $a_2(1) = 1$ and $a_2(0) = 0$ then a type τ that plays the action 1 regardless of signal has

$$\mu_{\tau|N} \leq \frac{8}{\left(1 - 4\left(\frac{\epsilon}{\pi}\right)\right)\underline{\mu}}\left(\frac{\epsilon}{\pi}\right).$$

Proof. When the long-run action a_1 does not depend upon the signal from Lemma 3

$$\mu_{z|\tau} = 1(a_1(\tau) = z)\sum_y \left[1(a_2(y)=1) + \pi 1(a_2(y)=0)\right]\mu_{y|\tau} + \epsilon H_{z\tau}.$$

Since $1(a_1(\tau) = -a_1) = 0$ it follows that $\mu_{-a_1|\tau} = \epsilon H_{z\tau} \leq \epsilon\overline{h}$. From Lemma 2

$\mu_{-a_1} \geq \pi\underline{\mu}$ and Bayes law then implies

$$\mu_{\tau|-a_1} \leq \frac{\epsilon 2}{\pi\underline{\mu}}.$$

For the second part we have from Lemma 3

$$\mu_{N|\tau} = (1-\pi)\sum_y [1(a_2(y)=0)]\,\mu_{y|\tau} + (1-\pi)\epsilon H_{N\tau}$$

$$\mu_{0|\tau} = \epsilon H_{0\tau}.$$

Hence

$$= (1-\pi)\left([1(a_2(0)=0)]\,\mu_{0|\tau} + [1(a_2(N)=0)]\,\mu_{N|\tau}\right) + (1-\pi)\epsilon H_{N\tau}.$$

Also from Lemma 3

$$\mu_{N|\tau} = (1-\pi)\,[1(a_2(N)=0)]\,\mu_{N|\tau} + (1-\pi)\,[1(a_2(0)=0)]\,\epsilon H_{0\tau} + (1-\pi)\epsilon H_{N\tau}$$

$$\mu_{N|\tau} \leq (1-\pi)\mu_{N|\tau} + (1-\pi)\epsilon H_{0\tau} + (1-\pi)\epsilon H_{N\tau}$$

so

$$\mu_{N|\tau} \leq \frac{(1-\pi)4\epsilon}{\pi}.$$

From Lemma 4

$$\mu_N \geq \frac{1-\pi}{2}\left(1-\frac{4\epsilon}{\pi}\right)\underline{\mu}.$$

Hence Bayes law implies

$$\mu_{\tau|N} \leq \frac{8\epsilon}{\pi\left(1-\frac{4\epsilon}{\pi}\right)\underline{\mu}}.$$

□

We define an *equilibrium response* of the short-run player to a strategy of the long-run player to be a best response to $\mu_{z\tau}$ induced by the long-run player strategy and itself.

Lemma 6. *For given V, Q there exists an $\underline{\epsilon} > 0$ such that for any ϵ satisfying*

$$\underline{\epsilon} > \frac{\epsilon}{\min\{\pi, 1-\pi\}} > 0$$

in any equilibrium the short-run player must enters on the good signal and only on the good signal. Moreover this is a strict equilibrium response.

Proof. We rule out all other possibilities

(a) A*lways enter $a_2(z) = 1$ for all z is not an equilibrium.* By Lemma 1 always enter implies no investment by the normal long-run player. By Lemma

5

$$\mu_{g|0} \leq \frac{2}{\underline{\mu}}\left(\frac{\epsilon}{\pi}\right).$$

Hence

$$\mu_{\{b,n\}|0} \geq 1 - \frac{2}{\underline{\mu}}\left(\frac{\epsilon}{\pi}\right)$$

so that short-run optimality and ϵ/π small requires $a_2(0) = 0$ a contradiction.

(b) *The unique equilibrium response to never invest is to enter only on $z = 1$.* From Lemma 5 never invest implies

$$\mu_{n|1} \leq \frac{2}{\underline{\mu}}\left(\frac{\epsilon}{\pi}\right)$$

and the same inequality holds for $\mu_{b|1}$. Hence

$$\mu_{g|1} \geq 1 - \frac{4}{\underline{\mu}}\left(\frac{\epsilon}{\pi}\right)$$

and for small ϵ/π this forces $\alpha_2(1) = 1$. By the same Lemma

$$\mu_{g|0} \leq \frac{2}{\underline{\mu}}\left(\frac{\epsilon}{\pi}\right)$$

for for small ϵ/π we must have $\alpha_2(0) = 0$.

We may now apply Lemma 5 again to conclude that

$$\mu_{g|N} \leq \frac{8}{\left(1 - 2\overline{h}\left(\frac{\epsilon}{\pi}\right)\right)\underline{\mu}}\left(\frac{\epsilon}{\pi}\right).$$

so that for sufficiently small ϵ/π the short-run player must stay out on N as well. All these responses are strict.

(c) *A equilibrium response requires $a_2(1) = 1, a_2(0) = 0$.* Any other strategy satisfies $a_2(0) \geq a_2(1)$. From Lemma1 this implies no investment by the long-run player. Part (b) then forces $0 = a_2(0) < a_2(1) = 1$ a contradiction.

(d) *The unique equilibrium response to always invest is to enter only on $z = 1$.* From Lemma 5

$$\mu_{g|0}, \mu_{n|0} \leq \frac{2}{\underline{\mu}}\left(\frac{\epsilon}{\pi}\right)$$

so

$$\mu_{b|0} \geq 1 - \frac{4}{\underline{\mu}}\left(\frac{\epsilon}{\pi}\right)$$

for small enough ϵ/σ implying $a_2(0) = 0$. Moreover

$$\mu_{b|1} \leq \frac{2}{\underline{\mu}}\left(\frac{\epsilon}{\pi}\right)$$

so that we have $a_2(1) = 1$. Apply We may now apply Lemma 5 again to conclude that

$$\mu_{g|N}, \mu_{n|N} \leq \frac{8}{\left(1 - 4\left(\frac{\epsilon}{\pi}\right)\right)\underline{\mu}} \left(\frac{\epsilon}{\pi}\right).$$

implying $a_2(N) = 0$. All these responses are strict.

This leaves only the strategy $\tilde{a}$ in which the long-run player plays $a_1 = 1$ on entry and $a_1 = 0$ if the short-run player stays out. As we known that $a_2(1) = 1, a_2(0) = 0$ there are two cases $a_2(N) = 1$ and $a_2(N) = 0$. First we must rule out the former.

We have entry on $N, 1$ and not on 0 consequently there is investment on $N, 1$ and not on 0. From Lemma 3 we find

$$\mu_{0|n} =$$

$$\sum_y 1(a_1(n,y) = 0)\left[1(a_2(y) = 1) + \pi 1(a_2(y) = 0)\right]\mu_{y|n} + \epsilon H_{0N} = \pi\mu_{0|n} + \epsilon H_{0n},$$

$$\mu_{N|n} = (1-\pi)\sum_y \left[1(a_2(y) = 0)\right]\mu_{y|n} + (1-\pi)\epsilon H_{Nn} = (1-\pi)\mu_{0|n} + (1-\pi)\epsilon H_{Nn}.$$

The former implies

$$\mu_{0|n} \leq \frac{2\epsilon}{1-\pi}$$

so that the second implies

$$\mu_{N|n} \leq 2\epsilon + (1-\pi)\epsilon H_{Nn} \leq 4\epsilon.$$

From Lemma 4

$$\mu_N \geq \frac{1-\pi}{2}\left(1 - \frac{4\epsilon}{\pi}\right)\underline{\mu}$$

so Bayes law gives

$$\mu_{n|N} \leq \frac{\pi}{1-\pi} \frac{8}{\left(1 - 4\left(\frac{\epsilon}{\pi}\right)\right)\underline{\mu}} \left(\frac{\epsilon}{\pi}\right).$$

Also by Lemma 5

$$\mu_{g|N} \leq \frac{8}{\left(1 - 4\left(\frac{\epsilon}{\pi}\right)\right)\underline{\mu}} \left(\frac{\epsilon}{\pi}\right).$$

Hence

$$\mu_{b|N} \geq 1 - \frac{16}{\left(1 - 4\left(\frac{\epsilon}{\pi}\right)\right)\underline{\mu}} \left(\frac{\epsilon}{1-\pi}\right)$$

from which the result follows. Note that is it only for this result that we require $\epsilon/(1-\pi)$ to be small as well as ϵ/π.

Finally we must show that $\alpha_2(N) = 0$ is in fact a strict equilibrium response for the short-run player. We have

$$\mu_{b|1}, \mu_{g|0} \leq \frac{2}{\underline{\mu}} \left(\frac{\epsilon}{\pi} \right)$$

$\mu_{b|1} = 0$ and $\mu_{g|0} = 0$ so is a strict best response to stay out in the former and enter in the latter. Finally Lemma 3 gives

$$\mu_{g|N} \leq \frac{8}{\left(1 - 4\left(\frac{\epsilon}{\pi}\right)\right) \underline{\mu}} \left(\frac{\epsilon}{\pi} \right)$$

implying for small ϵ/π it is strictly optimal for the short-run player to stay out on N.

□

Online Appendix: The Optimum of the Long-run Player

We examine the problem of the long-run player. Recall the Bellman equation

$$V(a_2) = \max_{a_1}(1-\delta)\left[a_2 - ca_1\right] + \delta \sum_{z'} P(z'|z, a_1)V(\alpha_2(z')).$$

We may write this out as

$$V(a_2) =$$

$$\max_{a_1}(1-\delta)\left[a_2 - ca_1\right] + \delta\left[(a_2 + (1-a_2)\pi) V(\alpha_2(a_1)) + (1-a_2)(1-\pi)V(\alpha_2(N))\right].$$

Lemma. [Lemma 1 in the Appendix] *The optimum for the long run-player is strict so pure and depends on the state only through* a_2*. The strategy* $\alpha_1(0) > \alpha_1(1)$*, is strictly sub-optimal. If* $\alpha_2(0) \geq \alpha_2(1)$ *it is strictly optimal for the long-run player never to invest, that is* $\alpha_1(z) = 0$*.*

Proof. The argmax is derived from:

$$\max_{a_1} -(1-\delta)ca_1 + \delta\left(a_2 + (1-a_2)\pi\right) V(\alpha_2(a_1)).$$

The gain to not investing is

$$G(a_2) = (1-\delta)c - \delta((a_2 + (1-a_2\pi)\left[V(\alpha_2(1)) - V(\alpha_2(0))\right].$$

It follows that not investing is strictly optimal if

$$V(\alpha_2(1)) - V(\alpha_2(0)) < \frac{1-\delta}{\delta\left(a_2 + (1-a_2)\pi\right)}c$$

and conversely.

The RHS is decreasing function of a_2. Hence if it is optimal not to invest when the short-run player enters, it is strictly optimal not to invest when the short-run player stays out. If it is optimal to invest when when the short-run player stays out then it is strictly optimal to invest when the short-run player enters. Hence The strategy $\alpha_1(0) > \alpha_1(1)$ is strictly sub-optimal as asserted

Suppose that $\alpha_2(1) = \alpha_2(0) = a_2$. Then

$$V(\alpha_2(1)) = V(\alpha_2(0)) =$$

$$\max_{a_1}(1-\delta)\left[a_2 - ca_1\right] + \delta\left[(a_2 + (1-a_2)\pi) V(a_2) + (1-a_2)(1-\pi)V(\alpha_2(N))\right].$$

implying that it is strictly optimal to choose $a_1 = 0$.

Suppose that $\alpha_2(0) = 1, \alpha_2(1) = 0$. Then in state 1 choosing $a_1 = 0$ gives 1 in very period while in state 0 the most attainable is 0 in the first period and 1 in very subsequent period. Hence $V(\alpha_2(1)) - V(\alpha_2(0)) > 0$ implying that not investing is strictly optimal in every state.

This covers the case $\alpha_2(0) \geq \alpha_2(1)$. It remains to show that for $\alpha_2(a_1) = a_1$ the best response of the long-run player is strict. If the response is not strict the condition for the gain to not investing must be zero

$$V(1) - V(0) = \frac{1-\delta}{\delta\left(a_2 + (1-a_2)\pi\right)}c.$$

Observe this cannot be the case at both states a_2.

(a) The tie is for $a_2 = 1$

In this case we have

$$V(1) = V(0) + \frac{1-\delta}{\delta}c$$

Moreover since $a_1 = 0$ must solve the Bellman equation for $a_2 = 1$ we have $V(1) = (1-\delta) + \delta V(0)$. Solving we find $V(0) = 1 - c/\delta$.

Since $a_1 = 0$ is optimal at $a_2 = 1$ it must be that $a_1 = 0$ is strictly optimal at $a_1 = 0$. Hence

$$V(0) = \delta\left[\pi V(0) + (1-\pi)V(\alpha_2(N))\right].$$

There are two sub-cases depending on whether $\alpha_2(N) = 0, 1$.

If $\alpha_2(N) = 0$ then $V(0) = \delta v(0)$ implies $V(0) = 0$. Since we previously found $V(0) = 1 - c/\delta$ this implies that $c = 1/\delta$ which is ruled out by generic cost.

If $\alpha_2(N) = 1$ then we have

$$V(0) = \delta\left[\pi V(0) + (1-\pi)V(0) + (1-\pi)\frac{1-\delta}{\delta}c\right]$$

which we solve to find

$$V(0) = (1-\pi)c/\delta.$$

Again this must also be equal to $1 - c/\delta$ so we have $(1-\pi)c/\delta = 1 - c/\delta$ or $c = \delta/(2-\pi)$ also ruled out by generic cost.

(b) The tie is for $a_2 = 0$

In this case we have

$$V(1) = V(0) + \frac{1-\delta}{\delta\pi}c.$$

Moreover since $a_1 = 1$ is optimal for $a_2 = 0$ it must also solve the Bellman equation for $a_2 = 1$, that is,

$$V(1) = (1-\delta)(1-c) + \delta V(1)$$

so that $V(1) = 1 - c$. Hence

$$V(0) + \frac{1-\delta}{\delta\pi}c = 1 - c,$$

or

$$V(0) = 1 - c - \frac{1-\delta}{\delta\pi}c.$$

Again, there are two sub-cases depending on whether $\alpha_2(N) = 0, 1$.

If $\alpha_2(N) = 0$ then again $V(0) = \delta v(0)$ implies $V(0) = 0$, giving

$$c\left[\frac{1-\delta+\delta\pi}{\delta\pi}\right] = 1$$

which is ruled out by generic cost.

If $\alpha_2(N) = 1$ since $a_1 = 0$ is optimal at $a_2 = 0$ wand $V(1) = 1 - c$

$$V(0) = \delta\left[\pi V(0) + (1-\pi)(1-c)\right]$$

or

$$V(0) = \frac{1-\pi}{1-\delta\pi}(1-c).$$

This must be equal to

$$1 - c - \frac{1-\delta}{\delta\pi}c$$

and equating the two we find

$$1 - c - \frac{1-\delta}{\delta\pi}c = \frac{1-\pi}{1-\delta\pi}(1-c)$$

$$c + \frac{1-\delta}{\delta\pi}c - \frac{1-\pi}{1-\delta\pi}c = 1 - \frac{1-\pi}{1-\delta\pi}$$

$$\frac{1-\delta}{\delta\pi}c + \frac{\pi-\delta\pi}{1-\delta\pi}c = \frac{\pi-\delta\pi}{1-\delta\pi}$$

$$(1-\delta\pi)(1-\delta)c + \delta\pi(\pi-\delta\pi)c = \delta\pi(\pi-\delta\pi)$$

$$c = \frac{\delta\pi(\pi-\delta\pi)}{(1-\delta\pi)(1-\delta)+\delta\pi(\pi-\delta\pi)}$$

ruled out by the generic cost assumption. □