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ESSAYS ON INFORMATION AGGREGATION AND MARKET CRASHES

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Economics

by

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DEDICATION

To my parents, my wife, and my two sons who have been so patient and loving.

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ABSTRACT OF THE DISSERTATION

ESSAYS ON INFORMATION AGGREGATION AND MARKET CRASHES

by

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This dissertation consists of three independent papers which address problems of information aggregation, market crashes, and equilibrium debt with bankruptcy, respectively.

Chapter 2 investigates the problem of information aggregation in a sequential action model. We defining an *informational cascade* as an event in which the sequence of actions converges to a limit and a *fully revealing informational cascade* as an event in which the sequence converges to a limit which is optimal under the true state. The necessary and sufficient condition for the occurrence of a fully revealing informational cascade *almost everywhere* is established.

Chapter 3 models market booms and crashes as an informational phenomenon in which a precipitous stock price drop is caused by a surprise which induces trading which reveals hidden information dispersed in the economy. We characterize the dynamics of market crashes through the following four phases: 1) confidence build-up, 2) mania, 3) trigger and 4) panic. We show that for some prior distributions on the states of nature, there exists a sequence of signals which generates

a price path that includes a market crash.

Chapter 4 investigates the long-term relationship between the borrower and the lender when there is a positive probability of bankruptcy. We define the "credible credit limit" as the debt level beyond which there is no loan with a positive expected return to the lender. Using this definition, characterizations of the time consistent equilibrium are provided, if they exist. The result indicates that a type of credit rationing may exist due to time inconsistency apart from asymmetric information or the unavailability of enforcement mechanisms.

Chapter 1

Overview

Chapter 2 investigates the problem of information aggregation in a sequential action model. A model in which agents sequentially take actions after observing the action history and private signal is constructed based on the one in Bikhchandani, Hirshleifer, and Welch(1991). An *informational cascade* is defined as an event in which the sequence of actions converges to a limit and a *fully revealing informational cascade* is defined as an event in which the sequence converges to a limit which is optimal under the true state; an informational cascade arises when agents take similar actions because they have posterior distributions close to each other and a fully revealing informational cascade arises if agents can infer the true state from the history of action choices. We relate our definition of informational cascades to that of Bikhchandani *et al.* and show that the informational cascade in their sense always has a positive probability of being non-fully revealing. The necessary and sufficient condition for the occurrence of fully revealing informational cascades *almost everywhere* is established. The necessary and sufficient condition is interpreted using the characterization in Milgrom(1979) and McKelvey and Page(1986).

Chapter 3 explains market booms and crashes as an informational phenomenon in which a precipitous stock price drop was caused by a surprise followed by the revelation of hidden information dispersed in the economy. We show that the market may not be able to aggregate information correctly if the agents sequentially take actions and the choice set is not fine enough to distinguish small differences in the private information. In particular small indivisibility in the trade unit is sufficient to prevent accurate information aggregation even when the market as a whole has enough information to make correct inferences as to the state.

We illustrate the dynamics of market crashes through the following four phases: 1) confidence build-up, 2) mania, 3) trigger, and 4) panic. Each phase is characterized by different set of beliefs of the public and agents who strike trades in the corresponding phase. We show that for a set of prior distributions on the states of nature, there exists a sequence of signals which generates a price path that includes a market crash.

Chapter 4 investigates the long-term relationship between the borrower and the lender when there is a positive probability of bankruptcy. In the long-term relationship with a borrower, the lender has a time inconsistency problem because the lender cannot make a binding commitment with respect to the credit limits. We examine the consequences of the time inconsistency problem in a two-person game based on a model originally developed by Hellwig(1977).

In contrast to Hellwig who focused on the non-existence of time consistent courses of action, we concentrate on the characterization of the time consistent equilibrium when it exists. We define the "credible credit limit" as the debt level beyond which there is no loan with a positive expected return to the lender. Using this definition we characterize the equilibrium as follows. First, in the case where the lender has a positive expected return from the loan up to the credible credit limit, the subgame perfect equilibrium results in a unique outcome in which the lender always extends the loan up to the credible credit limit. Second, if the lender has a negative expected return from the loan up to the credible credit limit, there exists a continuum of equilibria which all result in borrowing which is strictly less than the credible credit limit. We interpret the results as indicating that a type of credit rationing may exist due to time inconsistency apart from asymmetric information or the unavailability of enforcement mechanisms.

Chapter 2

On the Convergence of Informational Cascades

2.1 Introduction

An action taken by an individual often influences the decision of others made with its knowledge. In extreme cases it may appear as if the decision-maker ignores his own private information in favor of the history of previous actions. Bikhchandani, Hirshleifer, and Welch(1991, hereafter BHW) explain this localized conformity of human behavior as an informational phenomenon. If each action is taken based on the private information as well as public information, the history of action choices reveals private information as to the underlying parameter of the decision problem. Because a long string of action choices conveys some information about many random draws of the private signal, the agent may place more weight on the public information available from the history of action choice.

An intriguing aspect of their result is that an infinite sequence of agents may be induced to take action which is not optimal if given the knowledge of the true state; observationally an infinite number of random drawings of a private signal does not necessarily reveal the true state. Moreover, even if the sequence of actions converges, the limit may be not only uninformative but also misleading because it can be the optimal action for an unrealized state.

The present paper generalizes the result of BHW to a multiple action setting and shows that the informational cascades in the sense of BHW always have a positive probability of being uninformative. Our major result establishes the necessary and sufficient condition for the correct inference of the true state.

We develop our model based on the one in BHW. An infinite sequence of agents take actions after observing a private signal and the history of past action choice. The model is formulated as a problem of statistical inference in which the objective of the agent is to minimize the mean squared error. The action set is represented by a subset of the real numbers allowing an infinite action set.

We define *informational cascades* as the convergence of the sequence of actions. The definition differs from that of BHW in that they require an identical action for any signal while only the convergence is required in our definition. We also define *fully revealing informational cascades* as the convergence of the action to the limit which is the optimal action under the true state.

Our first theorem shows that the informational cascades in the sense of BHW have a positive probability of being non-fully revealing. The second theorem establishes the necessary and sufficient condition to guarantee fully revealing informational cascades given any prior distribution; the condition requires an action set which can reveal the posterior of each agent in a one-to-one fashion. In particular, a discrete action set always allows a positive probability of non-fully revealing informational cascades while a connected interval containing optimal actions for all states suffices to rule it out.

We can explain the intuition of our results as follows. Given a prior distribution with most of the probability concentrated on a state, the posterior updated in view of the private signal is not different from the prior by much. If the agent has only a few actions available, he cannot fine-tune the action choice according to the posterior which is only slightly different from the prior. As a result his action choice will not be different from the previous one. Once this happens, the action choice has no role in adding information to the history of signal draws.

The action choice in the model has the dual role of revealing the private information as well as minimizing the mean squared error of the decision-maker. From the standpoint of information revelation, a sparse action set provides little means to convey the private information. Consequently the infinite sequence of private signals adds little to the updating of the posterior distribution and the whole sequence of agents may end up choosing a wrong action.

Our result exhibits a striking contrast to Banerjee(1990). He shows that non-fully revealing informational cascades (herd behavior in his terminology) arise with positive probability even for a continuum action set. His result crucially depends on the degeneracy of the model; for instance, a concave payoff function suffices to recover fully revealing informational cascades everywhere, which is an implication of the present paper. Hence his result is not robust.

We can interpret our result in terms of characterization derived elsewhere in the literature. In particular, an action set satisfying our sufficient condition enables the agent to *distinguish* one state from another in the sense of Milgrom(1979). An action set satisfying the condition also satisfies the condition of the *stochastic regularity* defined by McKelvey and Page(1986).

The model is formally developed in Section 2.2. Section 2.3 provides the definitions of informational cascades and fully revealing informational cascades to be used in the analysis. The necessary and sufficient condition is provided in Section 2.4. In Section 2.5 we interpret our condition using the characterization of Milgrom(1979) and McKelvey and Page(1986). Section 2.6 concludes with the paper.

2.2 Model

At the beginning, a state is drawn randomly from the set of feasible states and remains fixed throughout. Each agent sequentially take action only once based on the public information and a private signal to minimize the loss from the action choice. The public information consists of the history of all actions taken by predecessors. The private signal which contains information about the realized state is drawn randomly. It is observed only by the agent taking the action at that time. The draw of signal is made according to a conditionally independent and identical distribution given the state.

Initially agents have prior distribution over the states. After observing the history of actions and the private signal, the agent updates the prior distribution according to the Bayes' rule. His action choice is optimal with respect to the posterior distribution after the updating of the prior.

Formally our model is specified as follows. There are finite number of feasible states, $s = 1, \ldots, S$. There are two signal values, $x \in X = \{1, 0\}$. The states are distinguished only by the probability of signal x, that is, $p_{1s} = \text{Prob}(x = 1|s) = p_s$ Then denote $p_{0s} = \text{Prob}(x = 0|s) = 1 - p_s$. The signal carries information about the state because p_s differs for different states s. Notice that the probability of the signal $x = 1$ given state s also denotes the mean of the signal x given state s because $E[x|s] = 1p_s + 0(1 - p_s) = p_s$.

Agent n takes an action a from the set of feasible actions, $A \subset \Re$. The action set A is the same for all agents. The set of feasible actions may be finite or (countably or uncountably) infinite but is assumed compact to guarantee the existence of optimal action. All agents minimize the same loss function $l_s(a)$ which depends on the state and the action taken. The information set of agent n includes the history of actions, $h^n = (a^1, a^2, \ldots, a^{n-1})$ and the private signal, x^n. We use the convention that $h^1 = \emptyset$.

We write the prior distribution given history h^n before the signal as $\mu_s^n = \mu(s|h^n)$ and the posterior distribution given history h^n and the private signal x^n as $\pi_s^n = \pi(s|h^n, x^n)$. We also write μ^n and π^n to denote $(\mu_1^n, \mu_2^n, \ldots, \mu_S^n)$ and $(\pi_1^n, \pi_2^n, \ldots, \pi_S^n)$, respectively. The structure of the model including the probability of the signal values given each state and the initial prior distribution over states, μ^1, is common knowledge.

We follow the notational convention that the superscript n corresponds to the

n^{th} agent while the subscripts s and x denote the s^{th} state and the signal x, respectively.

Agent n solves the following problem:

$$\min_{a \in A} E[l_s(a)|h^n, x^n], \tag{2.1}$$

or equivalently,

$$\min_{a \in A} \sum_{s=1}^{S} \pi_s^n l_s(a). \tag{2.2}$$

We make a set of assumptions to simplify the analysis.

Assumption 1 *The loss function is written as* $l_s(a) = (a - E[x|s])^2 = (a - p_s)^2$.

The assumption simplifies the notation substantially without loss of generality. It is straightforward to extend our result to a general setting under the standard regularity condition including the concavity of utility function.[1]

Assumption 2 *The probability of the signal,* $x = 1$, *is strictly between 0 and 1, and increases with the state* s :

$$0 < p_1 < p_2 < \ldots < p_S < 1.$$

The first part of Assumption 2 rules out the degenerate case and the second part implies that the signal, $x = 1$, is observed more often from a higher state than a lower state. It is a sufficient condition for the informativeness of the signal. An assumption is made in BHW for the same reason. The condition here looks simpler because there are only two signal values possible in the model. It is easy to see that there is no loss of generality arising from this feature compared to a model with many signal values.

[1]For example, assume that agents maximize the utility function which is a monotone transform of the negative of the loss function assumed in Assumption 1.

Assumption 3 *The action set A contains the means of signal for all states:*

$$\{p_1, p_2, \ldots, p_S\} \subset A.$$

Assumption 3 combined with Assumption 1 implies that knowing the true state, the agent would take the action which equals the mean of the true state.

2.3 Informational Cascades

We define the *informational cascade* and the *fully revealing informational cascade.*

Definition 1 *An* **informational cascade** *arises if*

$$lim_{n\to\infty} argmin_a E[l_s(a)|h^n, x^n] = \overline{a} \in A. \tag{2.3}$$

Definition 2 *A* **fully revealing informational cascade** *arises if*

$$lim_{n\to\infty} argmin_a E[l_s(a)|h^n, x^n] = \hat{a}_{\tilde{s}} = argmin_a l_{\tilde{s}}(a) \tag{2.4}$$

where $\tilde{s}$ is the true state.

In words, an informational cascade arises when agents take similar actions because they have posterior distributions close to each other and a fully revealing informational cascade arises if agents can infer the true state from the history of action choices.

The definition of the informational cascades appears distinct from the the one in BHW; they make the definition in terms of whether the action choice reveals the private signal. The following proposition shows that their definition implies a constant action when informational cascades arise. Because our definition does not require a constant action for informational cascades to arise but only the convergence of the action choices, informational cascades in our sense arise if informational cascades in their sense arise.

Proposition 1 *If there exists an N and h^N such that for all $x^N \in X$,*

$$\hat{a}^N = argmin_a E[l_s(a)|h^N, x^N] = \overline{a} \in A,$$

then for all $n \geq N$,

$$\hat{a}^n = argmin_a E[l_s(a)|h^n, x^n] = \overline{a}.$$

Proof: Fix a history h^N, and consequently μ^N, such that the hypothesis holds true. It suffices to show that

$$\hat{a}^{N+1} = \text{argmin}_a E[l_s(a)|h^{N+1}, x^{N+1}] = \overline{a}.$$

Denoting the probability of N^{th} agent's taking action a^N at signal x as $q(a^N|x)$, the prior of the $N+1^{st}$ agent can be written as:

$$\begin{aligned} \mu(s|h^{N+1}) &= \mu(s|h^N, a^N) \\ &= \frac{\mu(s|h^N)\sum_{x\in X} p_{xs} q(a^N|x)}{\sum_{t=1}^{S} \mu(t|h^N)\sum_{x\in X} p_{xt} q(a^N|x)}. \end{aligned} \tag{2.5}$$

Under the hypothesis of the proposition, $q(a^N = \overline{a}|x) = 1$ for all $x \in X$. It follows that the prior of the $N+1^{st}$ agent is the same as that of the N^{th} agent because

$$\begin{aligned} \mu(s|h^{N+1}) &= \frac{\mu(s|h^N)\sum_{x\in X} p_{xs} \cdot 1}{\sum_{t=1}^{S} \mu(t|h^N)\sum_{x\in X} p_{xt} \cdot 1} \\ &= \frac{\mu(s|h^N)}{\sum_{t=1}^{S} \mu(t|h^N)} \\ &= \mu(s|h^N). \end{aligned} \tag{2.6}$$

Also the hypothesis implies that starting with the prior μ^N, the loss is minimized by choosing $\overline{a}$ for any signal $x \in X$ so that

$$\hat{a}^{N+1} = \text{argmin}_a E[l_s(a)|h^{N+1}, x^{N+1}] = \overline{a} \text{ for all } x^{N+1} \in X.$$

Since we can repeat the same argument for all $n \geq N$, the proof is complete.

∎

It is interesting to notice that the informational cascades in the sense of BHW have a positive probability of being non-fully revealing. The next theorem establishes this fact. Their proposition that informational cascades in their definition arise with probability 1 for an action set with two elements is equivalent to the claim that for such an action set there is a positive probability of the occurrence of non-fully revealing information cascades. Indeed it will be shown that any discrete action set allows a positive probability of the occurrence of non-fully revealing informational cascades. It follows that their result generalizes to settings with more than two actions without change.

Theorem 1 *$Prob\{|lim_{n\to\infty} argmin_a E[l_s(a)|h^n, x^n] - \hat{a}_{\bar{s}}| > 0\} > 0$, if and only if there exists an N and h^N such that for all $x^N \in X$,*

$$\hat{a}^N = argmin_a E[l_s(a)|h^N, x^N] = \bar{a} \in A.$$

Proof: If $\bar{a} \notin \{p_1, \ldots, p_S\}$, then it is obvious that

$$\text{Prob}\{|lim_{n\to\infty} \text{argmin}_a E[l_s(a)|h^n, x^n] - \hat{a}_{\bar{s}}| > 0\} > 0,$$

because $\hat{a}_{\bar{s}} \in \{p_1, \ldots, p_S\}$. Hence we assume that $\bar{a} \in \{p_1, \ldots, p_S\}$.

Define

$$\mathcal{E}_{N,\bar{a}} = \left\{ h^N \middle| \begin{array}{l} \text{argmin}_a E[l_s(a)|h^N, x^N] = \bar{a} \text{ for all } x^N \in X \text{ and} \\ \text{for all } n < N, \\ \text{argmin}_a E[l_s(a)|h^n, x^n = 1] \neq \text{argmin}_a E[l_s(a)|h^n, x^n = 0] \end{array} \right\}$$

and $\bar{s}$ as the state for which $\bar{a}$ is the optimal action, that is, $\bar{a} = \text{argmin}_a l_{\bar{s}}(a)$. If there exists an N and h^N such that for all $x^N \in X$,

$$\hat{a}^N = \text{argmin}_a E[l_s(a)|h^N, x^N] = \bar{a} \in A,$$

then Prob$\{\mathcal{E}_{N,\bar{a}}\} > 0$ for a finite N. Moreover, the joint probability of the event $\mathcal{E}_{N,\bar{a}}$ and the occurrence of any state is positive because all finite history event has a positive probability of occurrence under all states. That is,

$$\text{Prob}\{\mathcal{E}_{N,\bar{a}},\ s\} > 0 \ \text{ for all } \ s \in S.$$

By Proposition 1, when the event $\mathcal{E}_{N,\bar{a}}$ occurs, all agents $n \geq N$ will take $\bar{a}$ and consequently

$$lim_{n\to\infty}\text{argmin}_a E[l_s(a)|h^n, x^n] = \bar{a} \in A.$$

Since the optimal action choice is different for different states and all states have positive probability of being true given the event $\mathcal{E}_{N,\bar{a}}$, there is a positive probability that $\hat{a}_{\tilde{s}}$ is different from $\bar{a}$, that is,

$$\begin{aligned}\text{Prob}\{\bar{a} \neq \hat{a}_{\tilde{s}}|\mathcal{E}_{N,\bar{a}}\} &= \text{Prob}\{\tilde{s} \neq \bar{s}|\ \mathcal{E}_{N,\bar{a}}\} \\ &= \frac{\sum_{s\neq\tilde{s}}\text{Prob}\{\mathcal{E}_{N,\bar{a}},\ s\}}{\text{Prob}\{\mathcal{E}_{N,\bar{a}}\}} > 0.\end{aligned}$$

It follows that

$$\begin{aligned}&\text{Prob}\{|lim_{n\to\infty}\text{argmin}_a E[l_s(a)|h^n, x^n] - \hat{a}_{\tilde{s}}| > 0\} \\ &\geq \sum_{\bar{a}\in A}\sum_{N=1}^{\infty}\text{Prob}\{\bar{a} \neq \hat{a}_{\tilde{s}}|\mathcal{E}_{N,\bar{a}}\} > 0.\end{aligned}$$

The sufficient part is proved by contradiction. Suppose there is no finite N for which Prob$\{\mathcal{E}_{N,\bar{a}}\} > 0$. Then for all finite N,

$$\text{argmin}_a E[l_s(a)|h^N, x^N = 1] \neq \text{argmin}_a E[l_s(a)|h^N, x^N = 0].$$

Consequently the history of the action choice reveals the history of the private signal. It is left to show that the posterior conditional on the history of the private signal converges to the point mass concentrated on the true mean with probability 1, that is, if the true state is $\tilde{s}$,

$$\lim_{n\to\infty}\pi(s|h^n, x^n) = \begin{cases} 1 & \text{if } s = \tilde{s} \\ 0 & \text{otherwise}\end{cases}$$

The proof is a special case of the strong law of large numbers adapted to the convergence of posterior. We omit the proof because it is standard.[2]

Hence if there is no finite N for which $\text{Prob}\{\mathcal{E}_{N,\bar{a}}\} > 0$,

$$\text{Prob}\{|lim_{n\to\infty}\text{argmin}_a E[l_s(a)|h^n, x^n] - \hat{a}_{\bar{s}}| > 0\} = 0,$$

and the proof is complete. ∎

Since we are interested in the condition to guarantee fully revealing informational cascades, we need to relax the definition so that it includes the case in which the sequence of action choices only converges; otherwise informational cascades may be always non-fully revealing.

2.4 Necessary and Sufficient Condition

In this section we develop the necessary and sufficient condition to guarantee fully revealing informational cascades for any initial prior μ. We start with a lemma about the optimal action choice.

Lemma 1 *Given a posterior distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_S)$, the optimal action is:*

$$\hat{a} = argmin_{a\in A}|a - E_\pi p| \tag{2.7}$$

Proof: First suppose the action set is a connected interval so that the first order condition can be applied to Problem (1). The first order condition for minimizing the mean squared error is

$$\sum_{s=1}^{S} \pi_s(a - p_s) = 0. \tag{2.8}$$

[2]For a proof, see DeGroot(1970) p.202.

Therefore $\hat{a} = \sum_{s=1}^{S} \pi_s p_s = E_\pi p$.

Next consider the case where the action set is not a connected interval so that the first order condition is not applicable. It suffices to show that if $|a - E_\pi p| \leq |a' - E_\pi p|$, then $E_\pi(a-p)^2 \leq E_\pi(a'-p)^2$.

Notice that

$$\begin{aligned} E_\pi(a-p)^2 &= E_\pi[(a - E_\pi p) + (E_\pi p - p)]^2. \\ &= E_\pi(a - E_\pi p)^2 + 2E_\pi[(a - E_\pi p)(E_\pi p - p)] + E_\pi(E_\pi p - p)^2. \end{aligned} \tag{2.9}$$

The second term in the last line of the above equation is identically 0 and the last term does not depend on the action a. Hence if $|a - E_\pi p| \leq |a' - E_\pi p|$, then $E_\pi(a - E_\pi p)^2 \leq E_\pi(a' - E_\pi p)^2$, and consequently $E_\pi(a-p)^2 \leq E_\pi(a'-p)^2$, as is to be shown. ∎

Lemma 1 implies that the optimal action minimizes the distance to the conditional expectation. The next lemma establishes that the conditional expectation is higher when the private signal is the high signal than when it is the low one.

Lemma 2 *For all prior μ, $E[p|\mu, x=1] \geq E[p|\mu, x=0]$.*

Proof: Subtracting the right hand side of the above inequality from the left hand side yields:

$$\begin{aligned} & E[p|\mu, x=1] - E[p|\mu, x=0] \\ &= \frac{\sum_{t=1}^{S} \mu_t p_t^2}{\sum_{t=1}^{S} \mu_t p_t} - \frac{\sum_{t=1}^{S} \mu_t (1-p_t) p_t}{\sum_{t=1}^{S} \mu_t (1-p_t)} \\ &= \frac{\sum_{t=1}^{S} \mu_t (p_t - \sum_{t=1}^{S} \mu_t p_t)^2}{(\sum_{t=1}^{S} \mu_t p_t)(\sum_{t=1}^{S} \mu_t (1-p_t))} \geq 0. \end{aligned} \tag{2.10}$$

The proof is complete. ∎

Our major theorem provides the necessary and sufficient condition for the occurrence of fully revealing informational cascades with probability 1 given any priors. The theorem is stated as an inequality condition to be satisfied by each point in the action set. To do this, we imagine each point, v, in the action set as having gaps to its right and left; writing ϵ_v^r as the radius of the gap to the right of point v and ϵ_v^l as the radius of the gap to the left of point v, respectively, an interior point has both radii equal to 0. Given the notation, we can state the theorem. The theorem provides a convenient means to check whether an action set allows non-fully revealing informational cascades.

Theorem 2 *Fully revealing informational cascades arise with probability 1 for all prior μ, if and only if for all v in the action set A such that $v \in [p_{\check{s}}, p_{\check{s}+1}]$ for some $\check{s}$,*

either $\epsilon_v^r = \epsilon_v^l = 0$,

or at least one of the two gaps of v is strictly positive for which

$$\frac{p_{\check{s}}(p_{\check{s}} - v - \epsilon_v^r)}{(1 - p_{\check{s}})(p_{\check{s}} - v + \epsilon_v^l)} \leq \frac{p_{\check{s}+1}(p_{\check{s}+1} - v - \epsilon_v^r)}{(1 - p_{\check{s}+1})(p_{\check{s}+1} - v + \epsilon_v^l)}.$$

Proof: Lemma 1 and Lemma 2 imply that given a prior μ such that

$$v - \epsilon_v^l \leq E[p|\mu, x = 0] \leq E[p|\mu, x = 1] \leq v + \epsilon_v^r, \tag{2.11}$$

the optimal action is v regardless of the signal received by the agent. It follows from Theorem 1 that fully revealing informational cascades arise with probability 1 for all priors if and only if there is no prior satisfying the inequality (2.11) above. Suppose that $\epsilon_v^r = \epsilon_v^l = 0$. Then the optimal action cannot be v for both signals because two conditional expectations in the middle of inequality (2.11) are not identical. Hence suppose that at least one of the two radii is strictly positive. We rewrite the inequality (2.11) explicitly in terms of the prior μ and the conditional probability of signals under each state as follows:

$$v - \epsilon_v^l \leq \frac{\sum_{t=1}^S \mu_t(1 - p_t)p_t}{\sum_{t=1}^S \mu_t(1 - p_t)} \leq \frac{\sum_{t=1}^S \mu_t p_t^2}{\sum_{t=1}^S \mu_t p_t} \leq v + \epsilon_v^r \tag{2.12}$$

There is no prior satisfying the inequality (2.12) if the following system of inequalities does not have a solution:

$$(*)\begin{cases} \mu_1 \geq 0 \\ \quad\vdots \\ \mu_S \geq 0 \\ \sum_{t=1}^{S} \mu_t = 1 \\ \sum_{t=1}^{S} \mu_t(1-p_t)(p_t - v + \epsilon_v^l) \geq 0 \\ \sum_{t=1}^{S} \mu_t p_t(p_t - v - \epsilon_v^r) \leq 0. \end{cases}$$

We apply Farkas' Lemma to the above system of linear inequalities with $\lambda_t, t = 1, \ldots, S$ denoting the dual variables for the first S inequalities, and λ_{S+1} and λ_{S+2}, those for the equality and λ, that for the second to last inequality, respectively. Then the inequality system (*) does not have a solution if and only if the following simultaneous equation system has a non-negative solution:

$$\begin{cases} \lambda_1 + \lambda_{S+1} - \lambda_{S+2} + (1-p_1)(p_1 - v + \epsilon_v^l)\lambda = p_1(p_1 - v - \epsilon_v^r) \\ \quad\vdots \\ \lambda_S + \lambda_{S+1} - \lambda_{S+2} + (1-p_S)(p_S - v + \epsilon_v^l)\lambda = p_S(p_S - v - \epsilon_v^r) \\ \lambda_{S+1} - \lambda_{S+2} = 0. \end{cases} \tag{2.13}$$

The simultaneous equation system (2.13) has a non-negative solution if and only if there is $\lambda \geq 0$ such that

$$\begin{cases} \lambda_1 = p_1(p_1 - v - \epsilon_v^r) - (1-p_1)(p_1 - v + \epsilon_v^l)\lambda \geq 0 \\ \quad\vdots \\ \lambda_S = p_S(p_S - v - \epsilon_v^r) - (1-p_S)(p_S - v + \epsilon_v^l)\lambda \geq 0 \end{cases} \tag{2.14}$$

Since $p_t - v + \epsilon_v^l \begin{cases} > 0 & \text{for } t > \check{s} \\ \leq 0 & \text{for } t < \check{s}, \end{cases}$ $p_{\check{s}} - v + \epsilon_v^l < 0$ unless $v = p_{\check{s}}$, in which case $\epsilon_v^l = 0$,

and $\dfrac{p_t(p_t - v - \epsilon_v^r)}{(1-p_t)(p_t - v + \epsilon_v^l)}$ is increasing in p_t with discontinuity at $p_t = v - \epsilon_v^l$,

there exists non-negative λ satisfying (2.14) if

$$\frac{p_{\check{s}}(p_{\check{s}} - v - \epsilon_v^r)}{(1-p_{\check{s}})(p_{\check{s}} - v + \epsilon_v^l)} \leq \frac{p_{\check{s}+1}(p_{\check{s}+1} - v - \epsilon_v^r)}{(1-p_{\check{s}+1})(p_{\check{s}+1} - v + \epsilon_v^l)}.$$

Therefore the condition is the necessary and sufficient condition for fully revealing informational cascades and the proof is complete. ∎

There are a few corollaries following immediately from Theorem 2. If the action set is the whole interval $[p_1, p_S]$, then fully revealing informational cascades arise with probability 1 for all priors because it is the case in which $\epsilon_v^r = \epsilon_v^l = 0$, for all v in action set. On the other hand, if the action set is disconnected at any of the means p_s, then non-fully revealing informational cascades arise with a positive probability. We state these facts without proofs.

Corollary 1 *Fully revealing informational cascades arise with probability 1 for all initial prior μ, if the action set A contains the interval $[p_1, p_S]$:*

$$[p_1, p_S] \subset A.$$

Corollary 2 *If the action set does not contain an open interval between p_1 and p_S such that for some s, p_s is one of its end points, there are initial priors μ generating non-fully revealing informational cascades.*

Corollary 2 implies that a discrete action set always allows a positive probability of the occurrence of non-fully revealing informational cascades *a fortiori*.

2.5 Alternative Characterizations

The necessary and sufficient condition derived in Theorem 1 can be characterized by alternative conditions discussed elsewhere in the literature. Examining our condition in view of the alternative characterizations sheds lights on understanding the role of action set in our model.

Milgrom(1979) derived the necessary and sufficient condition for the convergence of competitive bids to the true value of the object in an auction model.

The condition requires that every state can be distinguished from other states using a signal which is the private information. He defines that "state s can be distinguished from state $\tilde{s}$ using signal a" if i) $P(s) = 0$ or ii) $P(\tilde{s}) > 0$ and

$$\inf_{E} \frac{P(a \in E|s)}{P(a \in E|\tilde{s})} = 0. \tag{2.15}$$

where E is an event.

In our context the signal in the above definition can be understood as the history of action choices and the private signal. The action set satisfying our condition provides a sequence of signals which enables the agent to distinguish a state from another. Therefore the posterior reveals the true state *a.e.*

Comparing his theorem with ours, we can notice an important distinction. The mode of convergence in his theorem is *in probability*, while we have *almost everywhere* convergence so that our result is stronger than his. The distinction comes from the fact that the strong law of large numbers used in our theorem effectively shows that condition (2.15) is satisfied with $\liminf$ instead of $\inf$. In turn we can apply the strong law of large numbers thanks to the sequential structure of our model; because the action is taken sequentially with the knowledge of the previous action history, each action choice is made with the knowledge of the previous information if the action set satisfies our condition. In other words information accumulates in our model which feature Milgrom's auction model does not have.

Alternatively we can interpret the occurrence of non-fully revealing information cascades using the condition derived in McKelvey and Page(1986). Their main theorem says that if the public information satisfies *stochastic regularity*, then agents share the same posterior at an event which is common knowledge. A function is defined stochastically regular if and only if it assigns different values to two distributions one of which stochastically dominates the other. If the action set satisfies the condition of Theorem 2, then the action history which is the public information in the model is stochastically regular using their terminology. Hence we can apply their theorem proving that a convergent sequence of action necessarily generates the same posterior among the agents when it is common knowledge.

2.6 Conclusion

The present paper analyzed the problem of information aggregation in a sequential action model. Our result characterized the action set which guarantees fully revealing informational cascades a.e.

The sequential action model in the paper exhibits a remarkable property in aggregating information; if the action set satisfies our condition, the action sequence converges to the optimum under the true state with probability 1. Also it is easy to see that fully revealing informational cascades will not be fragile to small perturbation because in fully revealing informational cascades a long string of action choices reveals a long string of signal draws which is not shattered by a small new information.

Our result has an interesting implication for the claim that informational cascades characterizes the initial public offerings (IPO) of securities. In IPO, variable-price sale is prohibited by the SEC. It has the effect of promoting non-fully revealing informational cascades because buyers have a restricted action set as a consequence. If the issuer is better off with less information available to the purchasing public, the issuer would not be against the fixed-price sale which promotes non-fully revealing informational cascades.[3]

[3]See Welch(1991) for an interesting discussion of the phenomenon.

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Chapter 3

Market Crashes and Informational Cascades

3.1 Introduction

Historically, there have been a few big security price falls which have occurred over a short period of time. These events, termed market crashes, pose a challenging puzzle to economists because in the absence of substantial change in the state of nature, rational economic behavior does not predict such sudden changes in the price. More specifically there are two apparently contradictory aspects in the phenomenon, first the optimistic movement of the market before the crash and second the pessimism which is revealed to have existed.

This paper explains the market crash as a failure in the information aggregation in the security market due to small friction in the trading procedure, that is, the indivisibility of trade unit. Many small errors due to small friction in the trading procedure leads to a big crash through the mechanism of informational cascades.

Informational cascades are phenomena where a sequence of agents take similar actions as they try to exploit the information available from the history of previous action choices. When a sparse choice set is combined with the sequential structure of decision making, the history of action choices may not reveal private information and the market fails to arrive at the right price. If a small further information arrival can make the agents suspicious of the previous history, the informational cascades may unwind to create a market crash.

We construct a model in which information is dispersed throughout the economy in the form of private signals and each agent makes an investment decision based on his own private information and the history of previous agents' decision. Because the price of the security is set by a market maker equal to the expected fundamental value of the security conditional on the history of trade orders, the public belief conditional on the history is reflected in the price. If there is a sudden change in the public belief, then the price may change drastically which is described as the market crash.

We illustrate the dynamics of market crashes through four phases: 1) confidence build-up, 2) mania, 3) trigger, and 4) panic. In the illustration of the market crash through to the four phases, we focus on how the market may fail to aggregate information before crash although correct information is available to the market as a whole.

Each phase is characterized by sets of public beliefs prevailing at the beginning and at the end of the phase and the sequence of private signals underlying trade orders made in each phase. Our major theorem establishes that given initial public beliefs satisfying the condition for the first phase, the sequence of signals specified in the characterization generates the price path of a market crash.

An example is used to show how the sequence of private signals satisfying the characterization of the theorem generates the price path of market crash. An important distinction is made between the case where the price drops due to a bad state of the world and the case where the price drops due to a misleading signal in a good state of the world. In the bad state case, the price drop is permanent because there are many signals bearing on the bad state, while in the good state case the price bounces back soon as there are more signals indicating the good state. Also we compute the price path for a smaller indivisible trade unit to see the impact of the size of the indivisibility. We can confirm that the smaller the indivisibility is, the more likely it is that the market will correctly aggregate information.

There have been numerous attempts to explain the international market crash of October 1987. Most notably there have been attempts to find the source of the market crash in the trading strategy. Genotte and Leland (1990) and Jacklin, Kleidon, and Pfleiderer (1990) explain the market crash by the asymmetric information as to the amount of portfolio insurance. Jacklin and others focus on the issue of failure of information aggregation while Genotte and Leland focus on the discontinuity of price paths. Although they provide logically consistent explanations, the data of the 1987 market crash indicates that the amount of portfolio insurance was smaller than expected which is the opposite of their hypothesis. Moreover, the crash was observed in countries without portfolio insurance (Roll (1989)).

The rest of the paper is organized as follows. The next section explains the stylized facts of market crashes. It identifies the puzzle to be resolved and provide the intuition of our explanation. Section 3.3 formalizes the intuition in a model with a sequential trade structure and an indivisible trade unit. Section 3.4 characterizes the dynamics of market crashes and establishes that the characterization generates market crashes. An example simulates the model for a set of parameters. We discuss important features of our model and relate them to historical events in Section 3.5. Section 3.6 concludes.

3.2 Stylized Facts

Most market crashes are characterized by the following three stylized facts.

1. At the time of the market crash, no major event changing the state of nature happens.

2. Before the market crash, the price rises steadily for a substantial length of time.

3. After the market crash, the price remains low for a substantial length of time.

The first stylized fact is an essential feature of market crashes; without this feature there is nothing intriguing about these events because prices should change with a big change in the state of nature such as a war or a bad harvest. Under the presumption that the first stylized fact is valid, the second one and the third one seem to contradict rational expectations. Given enough time to reveal information, the market should have learned the correct underlying parameter of the security price so that such a big price fall should not occur and subsequently sustain for a long time. On the other hand if the price drop is triggered by a trading strategy independent of the state of nature, there is no reason why prices should not revert to a high level before long.

The puzzle we want to explain in the present paper is how information can remain hidden from the market when the collection of private information would provide correct information as to the state of nature. In particular we argue that market crashes can occur due to the failure in information aggregation even when agents are rational expected utility maximizers.

Our explanation of the market crash relies on the concept of informational cascades. If investors make investment decisions after observing the history of previous decisions, they would rationally attempt to exploit the information available from the history. Because the history provides information as to many investors' private signals, each individual places only a small weight to his own information. Therefore two agents making decisions in a row would make similar decisions because the difference in their private information is dominated by the same public information which they both take into account. If the choice set has only a few elements, then similar decisions may become identical. We call this phenomenon of similar action choice the informational cascades. It can be shown that the informational cascades can be uninformative and fragile if only such a sparse choice set is available (Bikhchandani, Hirshleifer, and Welch (1991) and Lee (1991)).

Using the concept of informational cascades the market crash is modelled as an event where the uninformative and fragile cascade is reversed by the revelation of hidden information. The uninformativeness and fragility of the informational cascades in the model follows from the sparsity of the choice set. The indivisibility of trading order provides the reason why the choice set is not fine.

The indivisibility of trade unit in the model is one of the fundamental institutional structure of the security market. For instance, New York Stock Exchange (NYSE) has a round lot market and an odd lot market in addition to the market for block trading (upstairs market). The round lot market in which trade orders can be made only in the unit of 100 shares is responsible for most volume of trade. The odd lot market handles trade orders smaller than 100 shares but the trading volume amounts to a small portion of the total trading volume of NYSE. Therefore the informational flow would be mostly from the round lot market to odd lot market and not *vice versa*. Since the informativeness of the informational cascades depends on the market in which most information is revealed, the indivisible unit of trade in NYSE can be regarded as 100, although our result holds true for even smaller unit.

3.3 Security Market with Indivisible Unit of Trade

There is a risky security which pays its owner a state-contingent payoff, v_s, in the future. The security is traded in a security market to a sequence of agents who want to use it for smoothing the consumption between the present and future. The market has a market maker who deals with a sequence of agents. The market maker trades with only one agent in each round. He posts a price, p_n, to n^{th}

agent and fills any order, z_n, placed by the trader at the price. After each round he can update the price and he updates the price so that it equals the expected fundamental value of the security conditional on the information available from the history of previous trading orders.[1] The market maker always has enough inventory of the security and cash to satisfy any finite (short or long) order.[2] Trading is restricted to take place only at integer multiples of indivisible trade unit, d.

Initially a state pertaining to the fundamental value of the security is drawn randomly according to a prior distribution which is common knowledge in the market and stays fixed throughout. The market maker and traders do not know the realized state but each trader is given a private signal, θ, drawn independently according to a conditional probability distribution, $q_{\theta s}$, given a state s.

Agents are risk averse with an identical utility function which is additively separable over time. They allocate an identical initial wealth, W, to the riskless current consumption, c_n, and to the risky investment, z_n, which gives the future consumption. When they make the investment decision, they maximize the expected utility conditional on the information available to them. Each agent's information set, Ω_n, contains the history of prices and his own private signal, that is, $\Omega_n = \{p_1, \ldots, p_n; \theta_n\} = \{h_n; \theta_n\}$, where $h_n = \{p_1, \ldots, p_n\}$, and $h_0 = \emptyset$.

Agent n's investment problem is written as:

$$\max_{c_n, z_n \in Z} E[u(c_n) + \beta u(v z_n) | \Omega_n] \tag{3.1}$$

$$\text{s.t. } c_n + p_n z_n \leq W.$$

[1]Brennan and Thakor(1990) employ a similar model of the security market. In their model, the market maker does not take his own position but only crosses out corresponding trading order while the market maker in our model trades on his account.

[2]This simplifying feature of our model is not crucial in deriving the result. Alternatively we can work with changing inventory of security and cash after each trading round imposing *ad-hoc* allocation rules when the order cannot be filled with the inventory.

where β denotes the common discounting factor, and Z is the set of integer multiples of the indivisible trade unit, d.

In the following we use z_n to denote the optimal solution to problem (3.1) and x_n to denote the solution to problem (3.1) without the restriction on the trade unit. We make a few assumptions to simplify the analysis and notations.

Assumption 1 *Agents have the power utility function and the discounting factor of 1:*

$$u(c) = \frac{c^\gamma}{\gamma}, \quad \gamma < 1,$$
$$\beta = 1.$$

The power utility function has the constant relative risk aversion of $1-\gamma$. This feature makes our result robust to scale changes in the initial wealth and the indivisible trade unit so that only the relative magnitude matters in the analysis.

Assumption 2 *There are two states with increasing fundamental values of the security, $v_B < v_G$, and the initial prior probability distribution of the states, (μ_B^0, μ_G^0), is non-denegerate:*

$$0 < \mu_B^0, \mu_G^0 < 1.$$

The second part of Assumption 2 implies that each state, $s = B$ (denoted by B meaning "Bad") or $s = G$, (denoted by G meaning "Good") has a strictly positive probability of realization. Because there are only two states, we write $\mu_G^n = \mu^n$ and $\mu_B^n = 1 - \mu^n$ where the superscript indicates that the beliefs are those of n^{th} agent. Similarly we use the notation $\pi_G^n = \pi^n$ and $\pi_B^n = 1 - \pi^n$.

Assumption 3 *There are 4 signals whose probability distribution conditional on states is non-degenerate and satisfies the strict monotone likelihood ratio property (MLRP):*

$$\text{for } s = B, G \text{ and } \theta = 1, 2, 3, 4,$$

$$1 > q_{\theta s} > 0 \text{ such that}$$

$$\frac{q_{1B}}{q_{1G}} > \frac{q_{2B}}{q_{2G}} > \frac{q_{3B}}{q_{3G}} > \frac{q_{4B}}{q_{4G}}$$

The monotone likelihood ratio property is a sufficient condition for the informativeness of the signal. We will alternatively call the four signals , $\theta = 1, 2,$ and 3 by R, L, M, and H standing for "Rare," "Low," "Medium," and "High," respectively.

Notice that the prior μ^n, the posterior π^n, and the optimal security purchase z_n are random variables whose values depend on the realization of the price history and the private signal. We often use the notation $\mu^n(h_n)$, $\pi^n(\Omega_n)$ and $z_n(\Omega_n)$ to indicate this functional relationship.

Lemma 1 *Let x_n denote the solution to problem (3.1) without the restriction on the trading unit. Under Assumptions 1-3, the n^{th} agent's optimal solution to the investment problem (3.1) without the restriction on the trading unit is given as:*

$$x_n = \frac{W p_n^{\frac{1}{\gamma-1}}}{p_n^{\frac{\gamma}{\gamma-1}} + [\sum_s \pi_s^n v_s^\gamma]^{\frac{1}{\gamma-1}}}.$$

Appendix contains the proof of the lemma.

Lemma 2 *Given any price history, h_n, or equivalently a prior, μ^n, the posteriors of the n^{th} agent are ranked by the private signals in the sense of the first order stochastic dominance:*

$$\pi^n(h_n; R) < \pi^n(h_n; L) < \pi^n(h_n; M) < \pi^n(h_n; H),$$

where the inequality ordering is equivalent to the ordering by the first order stochastic dominance because there are only two states.

Lemma 2 follows from Assumption 3 (MLRP) whose proof is provided in Milgrom (1981).

Lemma 3 *Given any history, h_n, or equivalently a prior, μ^n, the optimal security purchases both with the restriction and without the restriction, z_n and x_n, respectively, are increasing in the signal θ :*

$$z_n(h_n; R) \leq z_n(h_n; L) \leq z_n(h_n; M) \leq z_n(h_n; H),$$

and

$$x_n(h_n; R) < x_n(h_n; L) < x_n(h_n; M) < x_n(h_n; H).$$

Proof: We first show that x_n is increasing in the signal. Notice that $\sum_s \pi_s^n v_s^\gamma$ is increasing in θ by a well-known theorem in Milgrom (1981) because π^n conditional on different signal is ranked by the first order stochastic dominance from Lemma 2 and v^γ is an increasing function. The denominator in the expression for x_n in Lemma 1 is decreasing in θ because $[\sum_s \pi_s^n v_s^\gamma]^{\frac{1}{\gamma-1}}$ is decreasing in θ for $\gamma < 1$. Therefore x_n is increasing in θ.

The weak inequality for z_n in the above lemma is the consequence of the indivisibility of the trading unit. This weak inequality plays a crucial role in preventing the accurate information aggregation in the analysis. The detail will be provided in Theorem 1. ∎

3.4 Market Crashes

This section analyzes the dynamics of market crashes. We illustrate the crash as proceeding in the following four phases: 1) confidence build-up, 2) mania, 3) trigger, and 4) panic. Each phase is characterized by a set of beliefs prevailing in the public and a set of signals upon which trades take place. Our main theorem establishes that there exists a set of priors and a set of signals satisfying the characterization which generates a price path for the market crash.

Phase 1 *Confidence Build-up: Given a public prior μ^0 such that*

$$z(\mu^0; H) > z(\mu^0; M) \geq z(\mu^0; L) \geq z(\mu^0; R),$$

a sequence of agents with signal H make trading orders until the public belief μ^{CB} satisfies:

$$z(\mu^{CB}; H) = z(\mu^{CB}; M) > z(\mu^{CB}; L) \geq z(\mu^{CB}; R).$$

Initially consider a market with a dispersed prior such that agents with different private signals make different trade orders despite the indivisible unit of trade. In particular, it is necessary that the agent with the most favorable signal H can make distinguishable trading orders to change the public belief in the direction of optimism. Given such a public prior belief, a sequence of optimistic trade orders will bid up the price because the market maker sets the price equal to the expected fundamental value of the security. The rise in the security price reflects the movement of the public belief into optimism which describes how the market build up confidence from the trading orders.

This first phase pushes the market belief into optimism until the public belief places enough probability weight on the good state. The phase is completed when the trade order from the less optimistic signal M does not distinguish itself from the one of the signal H due to their similarity and the indivisibility of the unit of trade.

Phase 2 *Mania: Given a public prior μ^{CB} such that*

$$z(\mu^{CB}; H) = z(\mu^{CB}; M) > z(\mu^{CB}; L) \geq z(\mu^{CB}; R)$$

a sequence of agents with signals either H or M make trading orders until the public belief μ^M satisfies

$$z(\mu^M; H) = z(\mu^M; M) = z(\mu^M; L) > z(\mu^M; R).$$

Then a sequence of agents with signals either H or M or L make trading orders without violating the last strict inequality above.

In the mania phase agents may appear irrational although they are making rational choices. The phase starts with the public prior placing enough probability on the good state that even without the indivisibility of the trade unit, the trade order from the signal M is close to the one from the signal H. Hence the trade orders will not be distinguishable under the restriction of the indivisible trade unit. An important consequence of this non-distinguishability is that no matter how many traders make trading orders with the signal M, they do not reveal themselves as such in the market but only contribute to push the public belief in the direction of optimism.

As the agents with signals either H or M make trading orders, the price will go up further as the expected fundamental value of the security goes up due to the higher probability of the good state inferred from the trading order. If this sequence is long enough, even traders with the skeptical signal L come to place great probability weight on the good state. As these traders join the market, the price remains high because their trading orders do not distinguish themselves due to the similarity of the trading orders and the indivisibility of the trade unit. Consequently the market fails to correctly aggregate the information available to the agents in the market.

Phase 3 *Trigger: Given a public prior μ^M satisfying*

$$z(\mu^M; H) = z(\mu^M; M) = z(\mu^M; L) > z(\mu^M; R),$$

an agent with signal R make a trading order resulting in the public belief satisfying

$$z(\mu^T; H) = z(\mu^T; M) > z(\mu^T; L) \geq z(\mu^T; R).$$

Imagine a signal which is observed only rarely but implies the bad state very strongly once observed. In particular we do not observe many traders with this signal. Assume that the probability of bad state conditional upon the observation of such a signal is much bigger than that of the good state. If this signal is to reveal a new information, the market belief should not be concentrated too much on the good state that even the rare signal cannot make a distinguishable trading order. However, given any non-degenerate prior, we can always find a conditional distribution so that the trade order conditional on R can be distinguishable from others. The information of this rare signal drives down the price because the market maker adjusts the expected value of the fundamental taking account of the big conditional probability of the bad state. If the price drops enough to make the trade orders from the signal L distinguishable, the market is ready for the next phase of panic.

Phase 4 *Panic: Given a public prior μ^T such that*

$$z(\mu^T; H) = z(\mu^T; M) > z(\mu^T; L) \geq z(\mu^T; R)$$

a sequence of agents with signal L make trading orders.

Given the prior which distinguishes the trade orders from the signal L, agents with such a signal reveal themselves as they make trading orders. If the state is indeed bad, it is more likely that more traders have the signal L so that price drops further following the initial drop in the trigger phase. It is important to notice that agents with the signal L could not be distinguished before trigger so that they make the same choice as those with the more optimistic signal while after the trigger they assess the probability of the bad state big enough to make distinguishable choices. It is conceivable that the price drops very rapidly because there are many agents who took the optimistic position despite their pessimistic signal in the mania phase and consequently try to unwind their position at the surprising news.[3]

The characterization of the last phase makes it clear that once a price drop is triggered by the rare signal, the price would drop further if the true state is bad, because under the bad state there are more agents with the signal L. In contrast, when the true state is good, the price drop would be temporary even if it is triggered by the rare signal because it cannot be supported by many agents with the signal L.

Our main theorem establishes that given a prior μ^0 satisfying the initial condition of Phase 1, the sequence of trading orders specified in each phase generates the sequence of public beliefs characterized above. First we introduce a few notations for the history of signals underlying the price history. $\{\theta\}^N$ denotes a history of prices possible from the sequence of the signal θ of length N, that is, $\{\theta\}^N = \{\theta, \dots, \theta\}$. There may be more than one argument inside the curly bracket in the case the same price history is possible from the sequence composed either of the signals inside the bracket. For example, $\{H, M\}^N$, denotes the price history when the traders with either H or M make N trades in a row. This history is possible when the market cannot distinguish two trade orders by the underlying signals because of the indivisible unit of trade. We begin with a few lemmas to be used in the proof of the theorem.

Lemma 4 *The likelihood ratios of states given a realization of signal satisfy the following condition:*

$$l_H < l_{HM} < l_{HML} < 1$$

$$\text{where } l_H = \frac{q_{HB}}{q_{HG}}, \quad l_{HM} = \frac{q_{HB} + q_{MB}}{q_{HG} + q_{MG}}, \quad \text{and} \quad l_{HML} = \frac{q_{HB} + q_{MB} + q_{LB}}{q_{HG} + q_{MG} + q_{LG}}.$$

[3]Due to the difficulty of designing the optimal strategy under the possibility of unwinding a position once taken, this scenario is formally assumed away in the analysis; we assume that each agent makes only one trade order in the whole game.

This is an easy consequence of the monotone likelihood ratio property. We omit the proof.

Lemma 5 *Given the initial prior μ^0, the public belief μ after the history $\{\theta\}^N$ is given by:*

$$\mu(\mu^0; \theta^N) = \frac{1}{1 - l_\theta^N(1 - 1/\mu^0)}$$

where l_θ denotes the likelihood ratio of states as in Lemma 4 given the signal(s) θ. Moreover, the public belief is increasing as the signal inferred from the trade order increases in its favorableness: given a non-degenerate prior, μ^0,

$$\mu(\mu^0; H) > \mu(\mu^0; \{H, M\}) > \mu(\mu^0; \{H, M, L\}) > \mu^0.$$

Proof: After observing the trade order, the market maker updates the belief by computing the probability of the trade order from each possible signal. Given the initial prior of μ^0, suppose the market maker inferred that the same trade order can be made from any signal in the set Θ. Then his posterior belief of the good state is computed by:

$$\mu(\mu^0; \{\theta \in \Theta\}) = \frac{\mu^0(\sum_{\theta\in\Theta} q_{\theta G})}{\mu^0(\sum_{\theta\in\Theta} q_{\theta G}) + (1 - \mu^0)(\sum_{\theta\in\Theta} q_{\theta B})}.$$

Dividing the numerator and the denominator of the right hand side by $\sum_{\theta\in\Theta} q_{\theta G}$ yields,

$$\mu(\mu^0; \{\theta \in \Theta\}) = \frac{\mu^0}{\mu^0 + (1 - \mu^0) l_{\theta\in\Theta}}. \tag{3.2}$$

The second part of the Lemma follows because the likelihood is ranked as in Lemma 4.

To prove the first part of the lemma, rewrite the updating equation (3.2) for n and $n-1$:

$$\mu^n = \mu(\mu^{n-1}; \theta) \tag{3.3}$$

$$= \frac{\mu^{n-1}}{\mu^{n-1} + (1-\mu^{n-1})l_\theta}$$
$$= \frac{1}{1 + (1/\mu^{n-1} - 1)l_\theta}.$$

Taking the reciprocal of both sides of equation (3.3), we get

$$\begin{aligned} 1/\mu^n &= 1 - l_\theta + l_\theta/\mu^{n-1} \\ &= (1-l_\theta)(1+\ldots+l_\theta^{n-1}) + l_\theta^n/\mu^0 \\ &= (1-l_\theta^n) + l_\theta^n/\mu^0 \\ &= 1 - l_\theta^n(1-1/\mu^0). \end{aligned} \tag{3.4}$$

and the lemma follows. ∎

In the following theorem, we assume that the wealth of the agents W is an integer multiple of $2v_G d$. The implication of the assumption is that if the true state is known to be G with certainty, the optimal decision with the restriction on the trade unit is the exact solution without the restriction, that is, $z_n = x_n$.[4]

Theorem 1 *1. There exist $\mu^T \geq \mu^M \geq \mu^{CB}$ such that for $\mu^M \geq \mu \geq \mu^{CB}$,*

$$z(\mu; H) = z(\mu; M) > z(\mu; L) \geq z(\mu; R)$$

and for $\mu^T \geq \mu \geq \mu^M$,

$$z(\mu; H) = z(\mu; M) = z(\mu; L) > z(\mu; R).$$

2. There exist finite integers N^{CB}, N^M, and N^T such that

$$\mu^{CB} = \mu(\mu^0; H^{N^{CB}}),$$

$$\mu^M = \mu(\mu^{CB}; \{H, M\}^{N^M}),$$

$$\text{and } \mu^T = \mu(\mu^M; \{H, M, L\}^{N^T}).$$

[4]This assumption does not affect the result but the proof is affected for a boundary case where the agent is indifferent between two points in the restricted domain for the trading order. The proof of this boundary case is not provided because the reasoning is almost identical to the general case.

The proof of the theorem uses a technical lemma. The proofs of the lemma and the theorem are provided in the appendix.

The theorem establishes that the sequence of signals satisfying the characterization generates a price path of market crash. The next example demonstrates the transition of the market crash for a set of parameters. In particular it shows how the informativeness of cascades depends on the size of indivisibility and underlying state.

Example

There are two states with fundamental values $\{30, 40\}$. Each trader has identical utility function with a constant relative risk aversion, $u(c) = \frac{c^\gamma}{\gamma}$, $\gamma = .5$. Other data contain:

$$W = 1,000,000,$$

$$q = [q_{\theta s}] = \begin{bmatrix} .0001, & .00001 \\ .7, & .45 \\ .15, & .25 \\ .15 - .0001 & .3 - .00001 \end{bmatrix},$$

$$\mu^0 = .5.$$

For comparison we compute the price path when the indivisible trade unit is given as 10 and when it is .5; the smaller the indivisibility is, the harder it is for uninformative cascades to arise.

We consider two cases, one for bad state and one for good state. The states are distinguished by the number of each signal in such a way that the frequency of each signal in the sample path is the approximately the same as the conditional distribution of the signal; for instance, the sample path of bad state has 70 % of signal L, 15 % of M, and 15 % of H, respectively. The signals are ordered symmetrically and at the center we place the signal R. The first half of the sample path is ordered so that the arrival of the private signal confirms the characterization of each phase, that is, signal H is placed at the start to be followed by M and then L.

The price path computed for the sample paths and trade units are illustrated in Figure 1 through Figure 4. The price path in each figure is drawn by the solid curve, while the step function below in dotted line describes the sample path of the signals upon which each trader make trade order. In Figures 1 and 2 which correspond to the bad state, there are less of traders with the signal H and M which is reflected in the short length of the first two steps. In Figures 3 and 4 for the good state, there are more traders with these favorable signals.

Figures 1 and 2 are distinguished by the size of the indivisible trade unit. Figure 1 has 10 shares as the minimum trade unit while Figure 2 has .5 shares. The same applies to Figure 3 and Figure 4.

In Figure 1 the market maintains a high price until the signal R arrives although there have been many traders with the signal L. This is possible because uninformative cascades develop early by a few H signals and the trade orders made by L signal traders are not distinguished as such due to the indivisible trade unit. In contrast Figure 2 shows that price falls as soon as the number of L signal reaches a substantial level so that the market detects the private information correctly.

Figures 3 and 4 exhibit a similar phenomenon. In this case the distinction lies in that the price drop after the signal R is very short for a smaller indivisibility. This can be explained by the fact that in Figure 4 with smaller indivisibility the market detects more signals bearing on the good state before the uninformative cascades develop so that small number of L signal does not reverse the price path. Therefore the smaller the indivisibility is, the less fragile the cascade is.

Comparison of Figure 1 and 3 shows another interesting feature of the model. In Figure 1, crash is permanent because there are more L signals reflecting the true state while Figure 3 has only a temporary price fall because more good news prevail in the market. Therefore the market crashes characterized by the stylized facts are likely due to informational problem.

3.5 Discussion

In the previous section, we characterized the market crash as a failure in aggregating information dispersed in the economy. Agents in the model are described as acting rationally based on all information available at the moment of investment decision. Nonetheless, they fail to aggregate their private information correctly because of the indivisibility of trade unit.

A difficulty of an explanation relying on indivisibility like ours lies in that although each agent may make an error due to the indivisibility, small errors may be canceled out at the equilibrium. The informational cascades phenomenon due to the indivisibility demonstrates that small errors may accumulate systematically instead of canceling out when agents move sequentially. In contrast, small errors would cancel out if all agents move at once and the market locates the equilibrium price based on the total information. Therefore the model suggests the possibility of a serious flaw in the informational role of the market when the market is made sequentially.

Our model does not derive from the usual story of informed trade and liquidity trader. Compared to those models, there is less informational asymmetry in the model, yet the market does not aggregate information correctly. In this sense we can say that our model is based on the minimal amount of informational asymmetry. On the other hand the indivisibility is a necessary condition for informational cascades in the model. Indeed without the indivisibility, even models with asymmetric information between informed traders and liquidity traders would not generate the sample path of price as in our model because although the informed traders want to choose the same strategy as liquidity traders, the liquidity traders do not have such incentive so that the true information would be reflected in the price eventually which is the standard result of the rational expectations model. Hence our result relies more on indivisibility of trade unit than asymmetric information.

An alternative reason why small errors may accumulate systematically can be provided by a degree of bounded rationality on the part of the market maker. The market maker has the role of interpreting private signals underlying trade orders. If the market maker has only a limited capability of making distinction between slightly different two trade orders, the price posted by the market maker may not reveal the information available from the trade orders, resulting in informational cascades.

The sequence of signals in the characterization may appear very restrictive; it is necessary that agents with favorable signals move first to set an optimistic tone for the market. However, the restrictiveness in the arrival order underscores the fact that the phenomenon would not be observed too often. Moreover the condition characterizing the trade orders in each phase can be relaxed to allow the price to evolve through ups and downs before going up high. It is worthwhile to note that as we have more ups and downs, more information would be revealed so that the uninformative cascades would not occur as easy as otherwise.

The transition of market crash illustrated in the model is standard among real world investors who do not subscribe to the efficient market hypothesis. They usually postulate the transition of market as "Accumulation," "Distribution," and "Liquidation" which correspond roughly to confidence build-up, mania, and panic in our model. The important difference is that the same pattern can be generated by rational behavior of market participants.

After the market crash of 1987, most attention was given to program trading, especially portfolio insurance. There have been many attempts to formalize the problematic aspect of the portfolio insurance. Notable are Genotte and Leland (1990) and Jacklin, Kleidon, and Pfleiderer (1990) which is closely related to Grossman (1988). Genotte and Leland focus on a different aspect of the problem, that is, they explain the discontinuous price fall during the market crash, while we are concerned with how all the information bearing upon the bad state can be hidden in the market before the market crash. Jacklin and others provide a different explanation for the failure of information aggregation; they follow Grossman to formulate the informational problem due to the portfolio insurance. According to their explanation, the price was maintained high before the crash because the market did not realize that the strong demand came from portfolio insurers. Although they succeed in generating a pattern similar to the market crash, the evidence about the amount of portfolio insurance during the market crash of 1987 indicates the opposite; after the market crash, the surprise was not that there were more portfolio insurers than expected, but there were less. Therefore their explanation is not consistent with the data. Indeed it is possible that the market has been relying on the portfolio insurance more than it should in maintaining high price.

Another weakness of the explanation relying on portfolio insurance is that the strategy of portfolio insurance is not endogenously derived. The theory of portfolio insurance is developed based on the assumption that the price follows a stochastic process independent of the strategy. But the model predicts that the price path may not be independent of the strategy so that the optimality of the portfolio insurance strategy cannot be internally established. In contrast our model is based on the optimizing behavior of agents so that we do not derive the result from any exogenously given behavioral rule.

An important observation is that a small information revelation can bring about a great consequence if the market cannot aggregate information correctly. Indeed it is only one trade order in the trigger phase that brings out the whole revelation of hidden information in the model.

The fact that the price fall began in the week before the market crash in 1987 can be explained by the result. When the rare signal is revealed, the price begins to fall and subsequently crash occurs only when substantial number of low signals are revealed to the market.

Our result accords well with the claims of many traders that they were skeptical during the price rise. Because many agents with low signal may have been induced to buy at high price ignoring their private information due to uninformative cascades, they would regret their purchase once the state is revealed.

3.6 Conclusion

This paper explains the market crash by the failure of information aggregation due to the indivisibility of trade unit. The result demonstrates that small errors due to small friction may systematically accumulate to a big blunder instead of canceling out one another.

Although it is hard to subject the theory under an empirical examination because the market crash is fundamentally a rare event, it does have some testable implications. First in a market of sequential structure like ours, the past history of price contains information not available from the current price. This is in contrast to the well-known efficient market hypothesis. In particular, there is a bigger probability of price reversion if the past price history exhibits a monotone trend in which it is likely that uninformative cascades has developed by a small number of early trade orders.

Second our theory has bearings on the price clustering which is an important empirical regularity. The real stock market has regulations on the price grid to prevent arbitrage attempts to exploit other trader's order by timing. Although there is a good reason underlying the regulation, the market cannot extract the private information continuously from the individual traders if the price change discontinuously. In turn the price may cluster more often because the uninformative cascades develop easily due to less information available from the price history.

Our model provides a policy implication in designing a more efficient capital market. By executing more orders at once, we can avoid the distortion of the individual investment decision due to the previous uninformative history. In particular the auction mechanism which is employed when a substantial imbalance exists in the market should be considered with more significance because it alleviate the problem from the sequential structure of the market.

Finally according to the theory, the market crash is not a completely avoidable event by ruling out a particular trading strategy as long as the market has small friction such as the indivisible trade unit. The future research should be focused on how to minimize the probability of its occurrence and the magnitude of its adverse effect, should it happen again.

3.7 Appendix

Proof of Lemma 1: After substituting for c_n by $W - p_n x_n$ from the constraint, the first order condition for the problem (3.1) is written as:

$$-p_n u'(W - p_n x_n) + \beta E[vu'(vx_n)|\Omega_n] = 0.$$

Using the fact that $u'(c) = c^{\gamma-1}$ and $\beta = 1$ from Assumption 1, we rewrite the condition as:

$$(W - p_n x_n)^{\gamma-1} = \frac{\sum_{s=1}^{2} \pi_s^n v_s (v_s x_n)^{\gamma-1}}{p_n}.$$

Because x_n does not depend on s, the condition is equivalent to

$$(W - p_n x_n)^{\gamma-1} = x_n^{\gamma-1} \frac{\sum_{s=1}^{2} \pi_s^n v_s^\gamma}{p_n}.$$

Raising both sides to the power of $\frac{1}{\gamma-1}$ and rearranging the terms yields the solution to the problem without the restriction on the trade unit. ∎

The proof of Theorem 1 relies on the following lemma.

Lemma 6 *Consider the following maximization problem parametrized by* $\mu \in M \subset \Re$ *:*

$$\max_{x \in X} f(x; \mu)$$

where $\frac{\partial^2 f(x;\mu)}{\partial x^2} < 0$ *and* $\frac{\partial^2 f(x;\mu)}{\partial \mu \partial x} < 0$ *for all* x*, for all* μ*.*
Suppose that the problem has interior solution for all $\mu \in M$*. For two parameter values* μ_1 *and* μ_2 *such that* $\mu_1 \geq \mu_2$*, the optimal solutions* x_1^* *and* x_2^* *satisfy :*

1. $x_1^* \leq x_2^*$,

2. *for some real numbers* a *and* b*, such that* $a \leq x_1^* \leq x_2^* \leq b$

$$\frac{f(x_1^*; \mu_1) - f(a; \mu_1)}{f(x_1^*; \mu_1) - f(b; \mu_1)} \leq \frac{f(x_2^*; \mu_2) - f(a; \mu_2)}{f(x_2^*; \mu_2) - f(b; \mu_2)},$$

3. *if, given a and b, there is a $\hat{\mu}$ for which $f(a;\hat{\mu}) = f(b;\hat{\mu})$, then*

   for all $\mu > \hat{\mu}$, the optimal solutions for μ and $\hat{\mu}$, satisfy $x^* < \hat{x}$ and $f(a;\mu) > f(b;\mu)$,

   for all $\mu < \hat{\mu}$, the optimal solutions for μ and $\hat{\mu}$, satisfy $x^* > \hat{x}$ and $f(a;\mu) < f(b;\mu)$.

4. *Given a different objective function $g(x;\mu)$ such that $\frac{\partial^2 f(x;\mu)}{\partial\mu\partial x} \geq \frac{\partial^2 g(x;\mu)}{\partial\mu\partial x}$ for all x, for all μ, $\hat{\mu}_f \leq \hat{\mu}_g$ where $\hat{\mu}$ is defined above.*

Proof: Because the first order condition for the maximization problem must be satisfied for all μ,

$$\frac{\partial^2 f(x;\mu)}{\partial\mu\partial x}d\mu + \frac{\partial^2 f(x;\mu)}{\partial x^2}dx = 0.$$

Therefore $\frac{dx^*}{d\mu} \leq 0$ and for $\mu_1 \geq \mu_2$, $x_1^* \leq x_2^*$ which is the first claim.

Notice that for $\mu_1 \geq \mu_2$, $\frac{\partial f(x;\mu_1)}{\partial x} \leq \frac{\partial f(x;\mu_2)}{\partial x}$ because the cross derivative is assumed negative. For a and b such that $a \leq x^* \leq b$,

$$f(a;\mu) = f(x^*;\mu) - \int_a^{x^*} \frac{\partial f(x;\mu)}{\partial x}dx$$

and

$$f(b;\mu) = f(x^*;\mu) + \int_{x^*}^b \frac{\partial f(x;\mu)}{\partial x}dx.$$

It follows that

$$\begin{aligned}
\frac{f(x_1^*;\mu_1)-f(a;\mu_1)}{f(x_1^*;\mu_1)-f(b;\mu_1)} &= \int_a^{x_1^*} \frac{\partial f(x;\mu_1)}{\partial x}dx/(-\int_{x_1^*}^b \frac{\partial f(x;\mu_1)}{\partial x}dx) \\
&\leq \int_a^{x_2^*} \frac{\partial f(x;\mu_2)}{\partial x}dx/(-\int_{x_2^*}^b \frac{\partial f(x;\mu_2)}{\partial x}dx) \\
&= \frac{f(x_2^*;\mu_2)-f(a;\mu_2)}{f(x_2^*;\mu_2)-f(b;\mu_2)}
\end{aligned}$$

and the second claim follows.

To prove the third claim, notice that for a $\hat{\mu}$ such that $f(a;\hat{\mu}) = f(b;\hat{\mu})$, if $\mu > \hat{\mu}$, then

$$\frac{f(\hat{x};\hat{\mu}) - f(a;\hat{\mu})}{f(\hat{x};\hat{\mu}) - f(b;\hat{\mu})} = 1 \geq \frac{f(x^*;\mu) - f(a;\mu)}{f(x^*;\mu) - f(b;\mu)},$$

and thus $f(a;\mu) \geq f(b;\mu)$. The reverse case for $\mu < \hat{\mu}$ can be proved the same way. Finally the last claim follows because at $\hat{\mu}_f$, $g(a;\hat{\mu}_f) \leq g(b;\hat{\mu}_f)$ and the proof is complete. ∎

Proof of Theorem 1: It is straightforward but tedious to show that the objective function in the agent's maximization problem satisfies the condition on its cross derivative as in Lemma 6 for the relevant domain. We omit its proof.

Denote the optimal investment amount when the true state is known as G with certainty as $\overline{z}$. Then $\overline{p} = v_G$ and thus

$$\overline{z} = \frac{W}{2v_G}$$

from Lemma 1 and the assumption that W is an integer multiple of $2v_G d$.

Using Lemma 6, we can show that there is μ^{CB} close to 1 such that the agent with the signal H or M is indifferent between $z_n = \frac{W}{2v_G}$ and $z_n = \frac{W}{2v_G} + d$ so that for all $\mu \geq \mu^{CB}$ the market cannot distinguish between traders with two different signals and the proof of the first claim in the first part of the theorem is complete.

For the same reason we can show that there is μ^M such that for all $\mu \geq \mu^M$, $\overline{z} = z_n(\mu;H) = z_n(\mu;L)$.

Setting μ^T as the belief such that for all $\mu \geq \mu^T$, $\overline{z} = z_n(\mu;H) = z_n(\mu;R)$, completes the proof of the first part of the theorem.

The second part of the theorem follows from Lemma 2. In particular, N^{CB}, N^M, and N^T can be computed using the updating equation given in Lemma 2. For instance, given μ^0 and μ^{CB}, N^{CB} is the smallest integer satisfying the inequality,

$$\mu^{CB} \geq \frac{1}{1 - l_\theta^{N^{CB}}(1 - 1/\mu^0)},$$

and the proof is complete. ∎

Figure 3.1: Sample Path of Signal and Price: Case 1

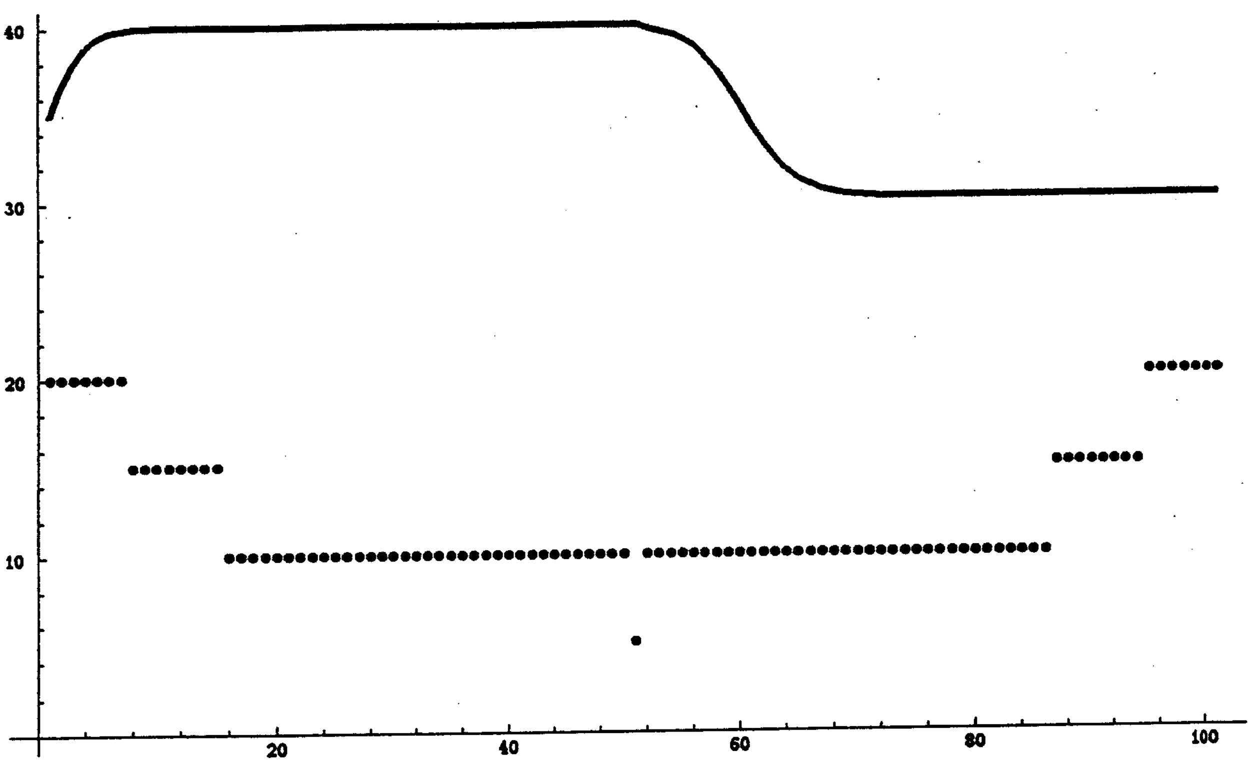


Figure 3.2: Sample Path of Signal and Price: Case 2

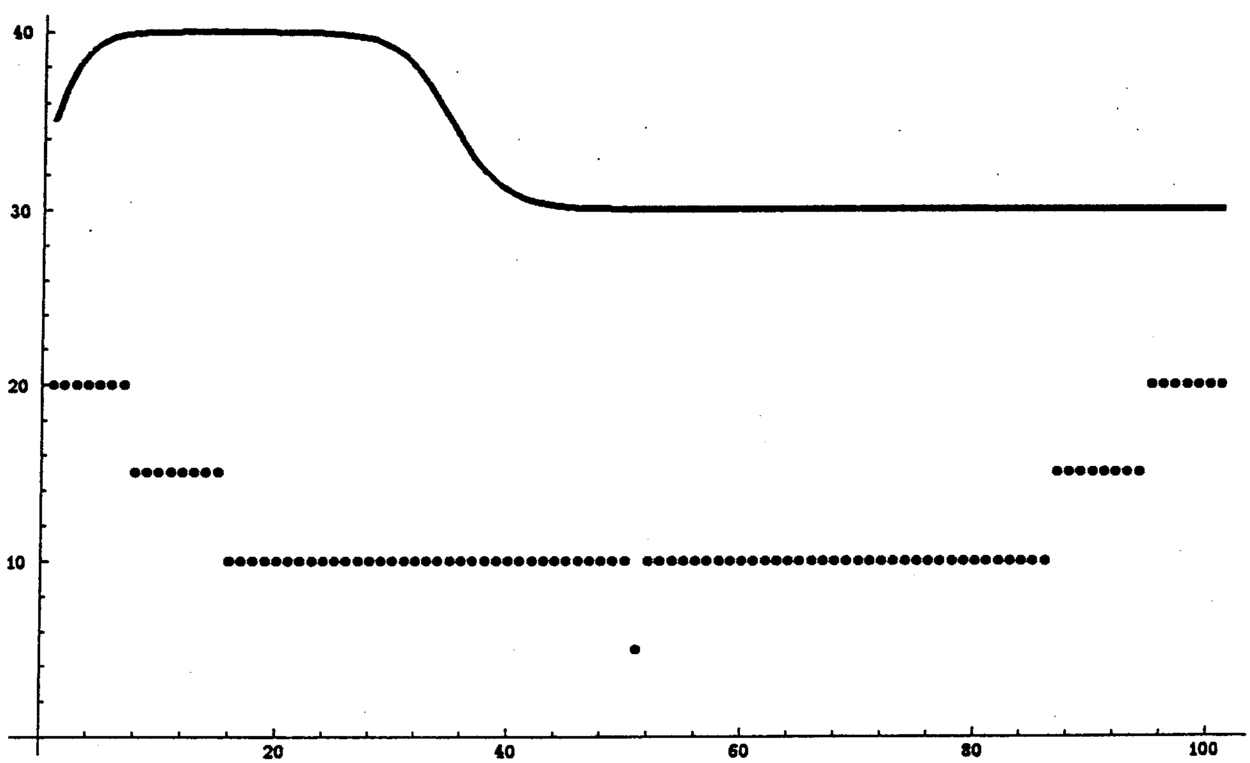


Figure 3.3: Sample Path of Signal and Price: Case 3

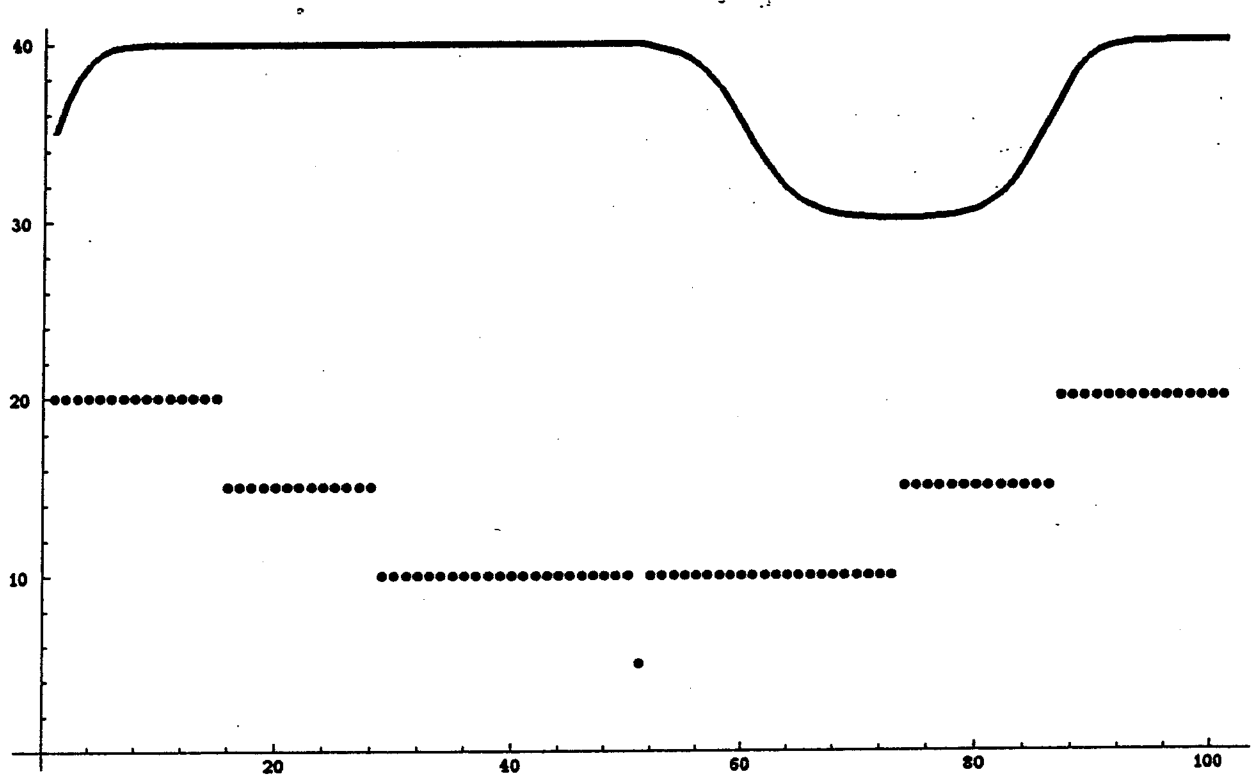
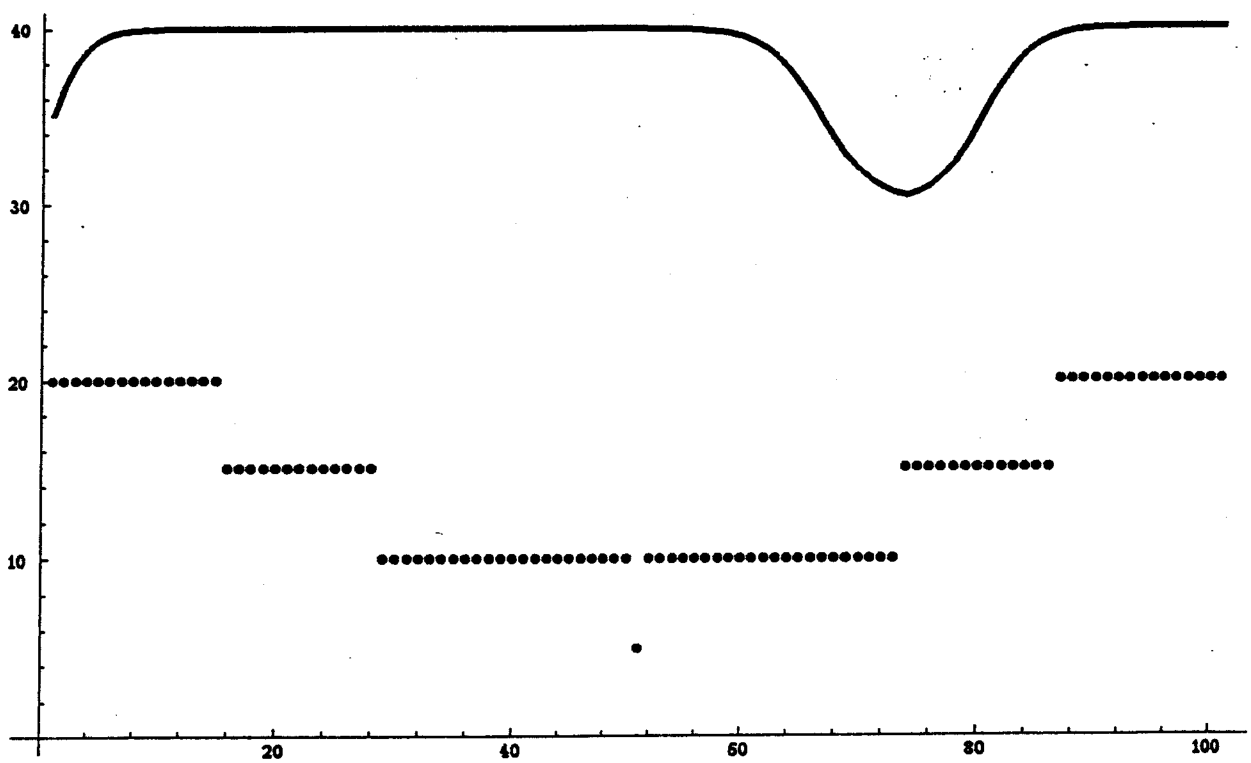


Figure 3.4: Sample Path of Signal and Price: Case 4



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Chapter 4

Equilibrium Borrowing and Lending with Bankruptcy

4.1 Introduction

Many economic models assume that an individual may borrow to smooth out the consumption path over time as far as it does not exceed one's total resources. Considering the pervasive presence of borrowing constraints in reality, however, the frictionless borrowing is obviously an extreme idealization. On the other hand, the literature on the borrowing problem usually seeks to explain the presence of borrowing constraints with asymmetric information or the unavailability of repayment enforcing mechanisms.[1] The present paper attempts to analyze the problem of borrowing and lending from a different perspective; in the long-term relationship with a borrower, a lender has a time inconsistency problem because the lender cannot make a binding commitment with respect to the credit limits. We explain the presence of borrowing constraints as a consequence of the time inconsistency problem of the lender.

Time inconsistency arises in the following way. Consider a borrower with uncertain future income. Specifically the borrower has no income initially but he may become rich making a constant income at an unknown time in the future. At the beginning of the relationship with the borrower, the lender may set a credit limit which maximizes his expected return if the borrower consumes subject to the given credit limit. If the borrower is unable to repay until after his borrowing hits the given credit limit, however, it may be renewed to a bigger one; otherwise the borrower has to go bankrupt immediately and the lender gets nothing back from his lending. Taking account of the anticipated increase in the credit limit, the borrower consumes more at each moment of time than he would subject to the outstanding credit limit. A higher consumption path shortens the time dur-

[1]See Bulow and Rogoff(1989a), Gale and Hellwig(1985), Green(1987), and Stiglitz and Weiss(1981), for such attempts.

ing which the given credit limit is exhausted and lowers the probability of the borrower's being able to repay while consuming the given credit. Consequently, the lender's expected return from the loan may become negative. Notice that the lender has the incentive to renew the outstanding credit limit at the time the borrowing reaches it even if the borrower has consumed subject to the additional credit limit.

From the explanation, it is immediate that the uncontrollability of the consumption path by the lender is a necessary element of the problem. Indeed, if the lender can control the borrower's consumption path, he can always guarantee himself a positive expected return by forcing the borrower to consume subject to the outstanding credit limit.

We analyze the problem in a game setting which is constructed using the model developed by Hellwig(1977). It is a partial equilibrium model where the interest rate is fixed at the market rate. It assumes symmetric information and the availability of a enforcement mechanism so that the borrower cannot default with enough money to repay the debt. The model is reformulated as a two-person game[2] where given a fixed interest rate, the lender maximizes the expected return from the loan to the borrower by setting a sequence of credit limits and the borrower maximizes the expected utility of consumption from borrowing with rational expectations as to the future evolution of credit limits.

The paper presents two major results. First, we provide a characterization of a class of the subgame perfect equilibria when they exist. The class of the subgame perfect equilibrium we characterize uses only pure strategies along the equilibrium path but most of them use mixed strategies off the equilibrium path.

To characterize the subgame perfect equilibrium, the "credible credit limit" is

[2]For an early game-theoretic approach to the time inconsistency problem, see Peleg and Yaari(1973).

defined as the debt level beyond which there is no loan with a positive expected return to the lender so that the lender stops lending regardless of the past history. We can identify two kinds of equilibria characterizations depending on whether the lender has a positive expected return when he initially sets the credit limit at the credible credit limit. In the first case where the expected return is positive, the lender always renews the credit limit up to the credible credit limit. The borrower ignores any credit limit less than the credible credit limit and consumes subject to the credible credit limit. In the second case where it is negative, the lender stops lending at a debt level less than the credible credit limit including the zero debt. The latter case implies that a type of credit rationing exists due to the time inconsistency problem of the lender.

The other result of the paper concerns the non-uniqueness of the equilibrium outcomes. In the second case of the equilibrium characterization, there is a continuum of equilibrium outcomes. The multiplicity of the equilibrium outcomes implies difficulties in predicting the equilibrium play of the lender and the borrower.

Our result can explain interesting situations including the sovereign debt problem. If it is postulated that the borrowing countries try to exploit the time inconsistent nature of the international banks, the frequent rescheduling of the sovereign debt services may be explained as equilibrium phenomena where the renewal of the credit limit occurs due to the changing incentive of the latter.

Before the formal analysis of the model, we briefly review the literature on the borrowing problem. Hellwig(1977) raised the issue of the time inconsistency of the lender from a different perspective. His paper was concerned with the non-existence of time consistent courses of action. In contrast to his paper we focus on the characterization of time consistent courses of action when they exist.

Moreover his result cannot be regarded as a theoretical justification of the presence of borrowing constraints because it predicts no existence of equilibrium.

Gale and Hellwig(1985) applied the game-theoretic reasoning to a one-period debt problem. They showed that the standard debt contract in reality emerges as the incentive-compatible debt contract. Their problem arises due to the informational asymmetry rather than the long-term relationship between the lender and the borrower. Hart and Moore(1989) analyzed the debt problem in a dynamic setting with renegotiation. The model assumes that enforcing repayment of debt is costly because the collateral is more valuable to the borrower than to the lender. Bulow and Rogoff(1989a) analyzed the sovereign debt problem focusing on the reputation effect of the borrower in the international capital market where the lender does not have enforcement mechanisms. These three models depart from the present one in that costly enforcement of repayment is the driving force of the problem.

Bulow and Rogoff(1989b) also studied the bargaining problem in rescheduling the sovereign debt services. Their bargaining equilibrium has a "one-time" nature so that it does not capture the implication of the long-term relationship between the borrower and the lender.

Chari and Kehoe(1989a,b) discussed the time inconsistency problem associated with the government debt in a different context from ours. The time inconsistency in the model arises because *de facto* it is efficient for the government to default.

Stiglitz and Weiss(1981) attempted to explain the credit rationing with asymmetric information as to the characteristics of the borrower. Their result differs from ours in that a borrower is either granted a loan or rejected totally.

To date, most of the literature has tried to explain the presence of borrowing constraints by postulating asymmetric information or the lack of enforcement mechanisms. In contrast, this paper explains borrowing constraints by the time inconsistency in the long-term relationship between the lender and the borrower and the uncontrollability of the borrower's consumption path.

The rest of the paper is organized as follows. In the next section we introduce our formal model with notations. In section 3 the model is formulated as a game. Section 4 contains the characterization and the non-uniqueness of the subgame perfect equilibrium when it exists. Section 5 concludes with comments on the possible directions of future research.

4.2 The Model

Consider a risk averse borrower who has no income or wealth currently but may become rich in the future. The borrower's objective is to maximize the expected utility by borrowing against his uncertain future income. The borrower discounts the future consumption at the constant discount rate δ. The borrower's utility function $u(c)$ is continuously differentiable, strictly increasing, strictly concave, and bounded above and below. The utility function has unbounded marginal utility at zero consumption, that is, $\lim_{c\to 0} u'(c) = \infty$.

The borrower's income is the only source of uncertainty in the model. The stochastic process of income, $y(t)$, is described as follows. At $t = 0$, the borrower has no income: $y(0) = 0$. Once the borrower becomes rich, he earns constant income, a, at each moment of time from then on, that is, for all t, t', with $t' \geq t$, $\text{Prob}\{y(t') = a \mid y(t) = a\} = 1$. If the borrower is poor at t, he becomes rich according to the Poisson arrival process with parameter λ, that is, for all t, t', with $t' \geq t$, $\text{Prob}\{y(t') = a \mid y(t) = 0\} = 1 - e^{-\lambda(t'-t)}$.

When the borrower becomes rich, he converts his risky debt into a riskless security because with certainty, his income is sufficient to pay the interest on the debt. The interest rate on the riskless debt is r. Hence after becoming rich, the borrower consumes the remainder of the income after the interest payment and the lender's return from the rich borrower is the discounted sum of riskless interest stream accrued to the debt outstanding at the moment the borrower becomes rich.

The borrower's income is a common knowledge and there is an institution which forces the rich borrower to pay the interest on his debt. Therefore the rich borrower may not go bankrupt consuming his whole income without paying the interest.

While poor, the borrower may borrow up to the credit limit, A, set by a risk neutral lender.[3] The debt, $k(t)$, accrues interest at an exogenously given risky interest rate, R, which is larger than r because of the positive probability of the bankruptcy. Once the debt hits the given credit limit, the lender decides whether to renew the credit limit to a bigger one. If the lender decides to renew the credit limit, the borrower may continue to borrow and consume as before.

If the lender decides not to renew, the borrower is forced into bankruptcy. The borrower also goes bankrupt if the interest exceeds the income even when he is rich. In addition, the borrower may choose to go bankrupt when he is poor. Once bankrupt, the borrower consumes nothing until he becomes rich so that he cannot smooth his consumption path by borrowing against his uncertain future income.

In addition, the borrower gets a bankruptcy penalty when he goes bankrupt. The bankruptcy penalty, $p(k)$, depends on the amount of the debt outstanding at the moment the bankruptcy occurs. The bankruptcy penalty is assumed to be more than enough to wipe out the gain from going bankrupt, that is, the expected utility of becoming rich with zero debt in the unknown future. On the other hand,

[3]The credit limit can be renewed over time so that it is a function of time. For notational convenience, however, the formal notation is written as if the lender is committed to one credit limit because at the equilibrium only one credit limit matters to both agents.

the lender gets nothing back from the bankrupt borrower.

Taking account of the penalty, we write the value of bankruptcy to the borrower, $B(k)$, as follows:[4]

$$B(k) = \frac{u(0) + \lambda \overline{V}(0) - p(k)}{\delta + \lambda}.$$

The first two terms in the numerator represent the expected utility of zero consumption until becoming rich and the expected utility of becoming rich with zero debt in the unknown future, respectively.

Because the penalty is enough to wipe out the gains from the bankruptcy, $p(k)$ satisfies

$$p(k) \geq \lambda(\overline{V}(0) - \overline{V}(k))$$

where $\overline{V}(k)$ is the utility of becoming rich with the debt of k. In addition, it is assumed that the value of bankruptcy is less sensitive to the amount of debt than the utility of becoming rich with the same amount of debt, that is, $B'(k) \leq \overline{V}'(k)$.[5]

We write the borrower's problem formally as:

$$\max_{c(t)} E \int_0^\infty e^{-\delta t} u(c(t))dt \tag{4.1}$$

$$s.t. \begin{cases} \dot{k}(t) = R(k(t)) + y(t) - c(t), \\ k(t) \geq -A, \; k(0) = 0, \\ c(t) \geq 0 \end{cases}$$

where the expectation in the objective function is taken with respect to the stochastic income, $y(t)$.[6]

Notice that the solution to the problem comprises two different consumption paths, one before becoming rich and the other after becoming rich. After becoming

[4]To derive $B(k)$, set $u(c(t)) = u(0)$ and $\overline{V}(k(t)) = \overline{V}(0)$ in the borrower's optimization problem (3) to be explained later.

[5]This technical assumption is not directly used in the analysis. We make the assumption to use Hellwig's result on the borrower's optimal choice.

[6]There is an abuse of notation in denoting the interest accrued on the debt as $R(k(t))$ because after becoming rich the interest rate is the riskless rate r.

rich, the borrower solves a maximization problem under certainty. If the borrower becomes rich with initial debt k_0 at time 0 given a credit limit of A, the borrower' problem is:

$$\overline{V}(k_0) = \max_{c(t)} \int_0^{\infty} e^{-\delta t} u(c(t)) dt \tag{4.2}$$

$$s.t. \begin{cases} \dot{k}(t) = rk(t) + a - c(t), \\ k(t) \geq -A, \ k(0) = k_0, \\ c(t) \geq 0. \end{cases}$$

Notice that the interest on the outstanding debt is accrued at the riskless rate because the borrower is rich. The value function $\overline{V}(k_0)$ is the indirect utility of becoming rich with the debt of k_0.

Using the value function of the maximization problem after becoming rich, the borrower's problem is rewritten as follows:

$$V(0, A) = \max_{c(t)} \int_0^{T} e^{-(\delta+\lambda)t} [u(c(t)) + \lambda \overline{V}(k(t))] dt + e^{-(\delta+\lambda)T} B(k(T)) \tag{4.3}$$

$$s.t. \begin{cases} \dot{k}(t) = Rk(t) - c(t), \\ k(t) \geq -A, \ k(0) = 0, \\ c(t) \geq 0 \end{cases}$$

where T is the time of bankruptcy.

The two terms inside the integral represent the utility from the current consumption and the expected utility of becoming rich with debt, respectively. It implies that before becoming rich, the borrower maximizes the expected utility taking account of the impact of the current consumption on the utility of becoming rich with debt. Hence the value of integral is the expected utility before going bankrupt. In addition, the borrower takes account of the value of bankruptcy represented by the last term in equation (4.3).

The time of bankruptcy, T, is a decision variable of the borrower. Because the debt grows at the risky interest rate, the poor borrower cannot delay the bankruptcy forever. The borrower plans to go bankrupt at the point of time when the cost of delaying the bankruptcy dominates the gain from it. When the borrower chooses the consumption path, he decides the time to go bankrupt also. If the borrower does not become rich before that time, he goes bankrupt as planned.

We define the value function $V(k_0, A)$ to be the borrower's incremental indirect utility when he consumes subject to the credit limit A given the initial debt of k_0. In a departure from standard terminology, we sometimes call $V(k_0, A)$ the borrower's problem in the same situation.

Next we formulate the lender's problem. At each point of time, $\lambda(-k(t))$ is the expected amount of debt that becomes a safe asset if the borrower becomes rich. In addition the lender has to provide a consumption loan if the borrower is not rich. To get the instantaneous expected return at time t, we subtract the amount of the consumption loan from the expected amount of debt that becomes safe asset. The lender has access to an unlimited amount of fund at a riskless interest rate, r, so that his cost of lending is assessed at the riskless interest rate. Equivalently he discounts the future return at the riskless interest rate. The lender's problem is written as:

$$P(0, A) = \max_A \int_0^T e^{-(r+\lambda)t}[\lambda(-k(t)) - c(t)]dt \qquad (4.4)$$

where $k(t), c(t)$, and T are solutions of $V(0, A)$ and thus functions of A.

The first term of the integrand represents the expected amount of debt that becomes a safe asset if the borrower becomes rich at time t. The second term is the consumption loan net of the interest at time t if he is not rich. The variables in the lender's objective function are implicitly functions of the credit limit A and thus the lender's objective function is maximized with respect to the credit limit, A.

Analogously to the borrower's case, $P(k_0, A)$ is defined to be the lender's incremental expected return when he sets the credit limit A to the borrower with the initial debt of k_0, or the lender's problem in the same situation.

The Poisson process of the stochastic income is stationary and independent of the borrower's action choice. It implies that the credit limit may not be set contingent on the borrower's past consumption choice which, for convenience, is assumed observable. The borrower's action affects only the consumption path. The lender's incentive to renew the outstanding credit limit depends on the amount of outstanding debt, but not the past consumption path because the past consumption path does not affect the future prospect of becoming rich. In particular, assuming that the lender cannot make a commitment with respect to the credit limits, a credit limit previously set may be renewed voluntarily by the lender when the debt reaches the credit limit. Consequently a contract contingent on the past consumption path is not viable. In contrast, if it is assumed that the borrower may affect the future income prospect by past action choice, a loan contract contingent on the borrower's past action choice becomes viable.[7]

It is assumed that the lender renews the credit limit to a bigger one only when the debt hits the outstanding credit limit. It implies that the lender cannot directly control the borrower's consumption path; once setting a credit limit, the lender moves only when the credit limit is exhausted.

The credit limit set by the lender is assumed not to exceed the discounted sum of the rich borrower's income, $A_{\max} = a/r : A \leq A_{\max}$. Implicit in the assumption is that the lender is not allowed to forgive a part of the debt. If the debt grows

[7]See Stiglitz and Weiss(1981) for this possibility.

bigger than $A_{\max}$, even the rich borrower cannot pay the interest. Because the borrower goes bankrupt if he cannot pay the interest out of his income, the lender should forgive a part of the debt to avoid the bankruptcy of the rich borrower. Hence the assumption is equivalent to the no forgiveness of debt.[8]

4.3 The Game, Γ

Next we formulate the model as a two person game, Γ, in the agent normal form where the players are the lender and borrower at different points of time; the first lender moves at the beginning of the game and the second lender moves when the loan reaches the first credit limit and so on, and similarly for the borrower.

We define the strategies of the players. The lender's strategy, $\{A_n\}_{n=1}^{\infty}$, and the borrower's strategy, $\{c_n(t)\}_{n=1}^{\infty}$, are a sequence of credit limits and a sequence of consumption paths, respectively. We consider only pure strategies along the equilibrium path. The players' strategies satisfy the Markov property along the equilibrium path: the lender's strategy depends on the amount of the debt outstanding at the moment of the credit limit renewal and the borrower's strategy depends on the amount of outstanding debt and the current credit limit. In particular, they do not depend on the whole past history of credit limits and consumption paths.

It is easy to see that the borrower's strategy does not depend on the past history because the borrower's problem $V(k_0, A)$ is defined given only the outstanding debt and the credit limit constraint. On the other hand, the lender's strategy depends only on the outstanding debt for the following reason. The previous credit limits may have affected at most the borrower's past consumption paths. But the borrower's past consumption paths do not affect the lender's

[8]Without the assumption, there may not be a subgame perfect equilibrium in the subgame after the debt reaches $A_{\max}$.

strategy choice because by the Poisson process of the stochastic income, the past consumption paths do not affect the prospect of future income. It follows that the lender's strategy depend on neither the previous history of credit limits nor the consumption paths.

The significance of the Markov property of the players' strategies lies in that a loan contract contingent on the past consumption choice is not viable. In particular, the lender's strategy is written as an increasing sequence which does not depend on the previous history.

The borrower's strategy is a sequence of continuous functions on time intervals because given the assumption on the borrower's utility function the solution to the problem $V(k_0, A)$ is a continuous function over time.

Let $A = (A_1, A_2, \ldots)$ denote the sequence of credit limits renewed at $t = T_0, T_1, \ldots$, and $c(t) = (c_1(t), c_2(t), \ldots)$ denote the sequence of consumption paths from T_0 to T_1, T_1 to T_2, and so on. The credit renewal times when the debt level reaches the outstanding credit limit, $T_n, n = 1, 2, \ldots$, are the borrower's decision variables. We are sloppy about this because they are also known, once the consumption decision is made. We denote the time the borrower begins borrowing by T_0 and thus $T_0 = 0$. The creditor's strategy space is the set of increasing sequences with supremum less than or equal to $A_{\max}$. The borrower's strategy space is the set of piecewise continuous functions on $[0, \infty]$.

In contrast to the case of strategies along the equilibrium path, we allow mixed strategies off the equilibrium path. The mixed strategies off the equilibrium path do not satisfy the Markov property; it depends on the previous credit limit the lender deviated to. The lender's deviation occurs when the lender renews to a credit limit which is not prescribed by the equilibrium strategy. To make the lender worse off after deviation, enough probability of loss should be warranted.

Because the expected return of the lender depends on the outstanding debt which is the previous credit limit renewed to by the deviation, the probability to make the lender worse off depends on the history of the credit limits. However, notice that the use of mixed strategy is never observed in the equilibrium play because it is used only off the equilibrium path.

Next we define the payoff functions of the players. The payoff functions have a recursive structure because the payoff at a certain moment of time is the sum of the payoffs from the contemporary play and from the future play. In the definition, we use the notation, $A_{-n} = (A_{n+1}, A_{n+2}, \ldots)$, and $c_{-n}(t) = (c_{n+1}(t), c_{n+2}(t), \ldots)$.

Definition 1 *The lender's payoff function, $\{P_n\}_{n=1}^{\infty}$, and the borrower's payoff function, $\{V_n\}_{n=1}^{\infty}$, are defined recursively by*

$$\begin{aligned} P_n(A_n, A_{-n}; c(t)) &= \int_{T_{n-1}}^{T_n} e^{-(r+\lambda)t}(\lambda(-k(t)) - c_n(t))dt \\ &\quad + e^{-(r+\lambda)(T_n - T_{n-1})} P_{n+1}(A_{n+1}, A_{-(n+1)}; c(t)) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} V_n(A; c_n(t), c_{-n}(t)) &= \int_{T_{n-1}}^{T_n} e^{-(\delta+\lambda)t}(u(c_n(t) + \lambda \overline{V}(k(t)))dt \\ &\quad + e^{-(\delta+\lambda)(T_n - T_{n-1})} V_{n+1}(A; c_{n+1}(t), c_{-(n+1)}(t)) \end{aligned} \tag{4.6}$$

$$s.t. \begin{cases} \dot{k}(t) = Rk(t) - c_n(t), \\ k(t) \geq -A_n, \;\; k(T_{n-1}) = -A_{n-1}, \\ c_n(t) \geq 0, \end{cases}$$

where $k(0) = A_0 = 0$, and

$$\text{if } \; A_{n-1} = A_n, \; \text{ then } \begin{cases} P_i(A_i, A_{-i}; c(t)) = 0 \text{ for all } i \geq n, \\ V_n(A; c_n(t), c_{-n}(t)) = B(-A_n), \\ V_i(A; c_i(t), c_{-i}(t)) = 0 \text{ for all } i \geq n+1. \end{cases}$$

In the definition above, the present payoffs, $P_n(\cdot;\cdot)$ and $V_n(\cdot;\cdot)$, depend on the future play via the future payoffs, $P_{n+1}(\cdot;\cdot)$ and $V_{n+1}(\cdot;\cdot)$, but not vice versa. This payoff structure at different moments of time points out the source of the time inconsistency problem.

In the analysis we use the subgame perfect equilibrium as our equilibrium concept. We first define the Nash equilibrium and then the subgame perfect equilibrium by requiring that the equilibrium strategy is a Nash equilibrium in all subgames. In the definition of the subgame perfect equilibrium, we denote the subgames starting from the n^{th} lender and the n^{th} borrower after the history A_{n-1} and A_n by $\Gamma^l_n|A_{n-1}$ and $\Gamma^b_n|A_n$, respectively. The strategy profiles $A|A_{n-1}$ and $c(t)|A_n$ are the projections of A and $c(t)$ on $\Gamma^l_n|A_{n-1}$ and $\Gamma^b_n|A_n$, respectively.

Definition 2 *$(\{A^*_n\}^{\infty}_{n=1}, \{c^*_n\}^{\infty}_{n=1})$ is a* <u>*Nash Equilibrium*</u> *if and only if*

$$1.\ P_n(A^*_n, A^*_{-n}; c^*(t)) \geq P_n(A_n, A^*_{-n}; c^*(t)) \tag{4.7}$$

for all A_n for all n,

$$2.\ V_n(A^*; c^*_n(t), c^*_{-n}(t)) \geq V_n(A^*; c_n(t), c^*_{-n}(t)) \tag{4.8}$$

for all $c_n(t)$ for all n.

Definition 3 *$(\{A^*_n\}^{\infty}_{n=1}, \{c^*_n\}^{\infty}_{n=1})$ is a* <u>*Subgame Perfect Equilibrium*</u> *if and only if*

1. for every possible history A_{n-1}, $(A^*|A_{n-1}, c^*(t)|A_n)$ is a Nash equilibrium for all $\Gamma^l_n|A_{n-1}$, $n = 1, 2, \ldots$,

2. for every possible history A_n, $(A^*|A_n, c^*(t)|A_n)$ is a Nash equilibrium for all $\Gamma^b_n|A_n$, $n = 1, 2, \ldots$.

The definition of subgame perfect equilibrium requires that the equilibrium strategy remains so in every subgame even after a history not prescribed by the equilibrium strategy.

4.4 Subgame Perfect Equilibrium

4.4.1 Characterization

The characterization of the subgame perfect equilibrium in Γ crucially depends on the credit limit beyond which there is no additional loan with positive expected return to the lender. Obviously the lender can precommit not to grant any loan in excess of this credit limit. We define this credit limit as the *credible credit limit.*[9]

Definition 4 *The credible credit limit, $\overline{A}$, is the smallest real number satisfying,*

$$\overline{A} = A_{\max},$$

or

$$P(-\overline{A}, A) < 0, \text{ for all } A \in (\overline{A}, A_{\max}].$$

By definition, for all $A < \overline{A}$, there always exits an additional credit limit $A' \in (A, \overline{A}]$ with positive expected return and $\overline{A}$ is unique.

On the other hand, there may not exist any $A < \overline{A}$ such that $P(-A, \overline{A}) \geq 0$, that is, for any $A < \overline{A}$, there exists $A' \in (A, \overline{A})$ such that $P(-A, A') \geq 0$ but $P(-A, \overline{A}) < 0$. If this holds true for the credible credit limit, then there is no subgame perfect equilibrium; the lender should renew if the additional credit limit is not to be renewed but the additional credit limit has the same property so that the lender cannot stop until the expected return becomes negative at $\overline{A}$.

[9]Our definition of the credible credit limit corresponds to that of the "naive cut-off point", A^N, in Hellwig's model.

Hellwig gives an example to show the non-existence of time consistent courses of action in the case above. Although Hellwig doesn't prove the non-existence of the subgame perfect equilibrium for his example, his non-existence result carries over to the game setting as alluded to above.[10] To avoid the non-existence of the subgame perfect equilibrium, we assume that the credible credit limit is locally profitable in the following sense.

Definition 5 *The credible credit limit is* <u>*locally profitable*</u> *if there exists* $k_0 \in (-\overline{A}, 0]$, *such that*

$$P(k_0, \overline{A}) \geq 0$$

The next lemma summarizes a few of Hellwig's results on the borrower's optimal consumption behavior to be used in our analysis. The numbers in the parentheses denote the proposition numbers used in Hellwig(1977).

Lemma 1 *(a)If* $r < \delta + \lambda$, *then the borrower wants to borrow for all* $k_0 \leq 0$. *(Prop.2(c))*

(b)For all k_0, $u'(c(t)) \geq \overline{V}'(k(t))$. *(Prop.4(1)(a))*

(c)The borrower chooses to go bankrupt only after exhausting the given credit limit. (Prop.4(2)(a))

We assume that $r < \delta + \lambda$ so that the borrower wants to borrow. Lemma 1(b) implies that given the same amount of debt, the consumption before becoming rich is smaller than after becoming rich. By Lemma 1(c), bankruptcy occurs only if the lender refuses to renew the credit limit or the debt grows to $A_{\max}$ in which case the income of the rich borrower is not enough to pay the interest.

[10]The non-existence of the subgame perfect equilibrium in the example is due to the continuity of the strategy space; the existence of the subgame perfect equilibrium is guaranteed only in a finite game.

The characterization of the subgame perfect equilibrium uses the backward induction starting from the credible credit limit. The lender's strategy given a certain amount of debt depends on whether a loan up to the credible credit limit yields a positive expected return and thus $P(k_0, \overline{A})$ as a function of k_0 plays an important role. We now proceed to a few preliminary results concerning the regularity of the lender's expected return $P(k_0, \overline{A})$ as a function of k_0 : the lender's expected return $P(k_0, \overline{A})$ is continuous in k_0 and it crosses zero downward at most once. All proofs are relegated to the appendix.

Lemma 2 *Given $A \geq 0$, $P(k_0, A)$ is continuous in $k_0 \in [-A, 0]$.*

Lemma 3 *If $-\frac{cu''(c)}{u'(c)} \geq \frac{R-\delta}{R+\lambda}$ for $c(t)$ such that $-\lambda k(t) - c(t) = 0$, then $P(k_0, A)$ is decreasing in k_0 when $P(k_0, A) \leq 0$.*

The condition of Lemma 3 guarantees that the consumption does not increase too fast relative to the debt. It requires that the relative risk aversion is greater than a certain number which depends on the parameters of the problem. The number would be fairly small in most cases so that most of well-behaved utility functions satisfy the condition. In the following, we assume that the condition of Lemma 3 holds.

The lender's expected return from the loan up to the credible credit limit is either positive or strictly negative. As will be shown later, we can draw $P(k_0, \overline{A})$ as a function of k_0 as in Figure 1 and Figure 2 depending on whether $P(0, \overline{A})$ is positive or negative.

The following example shows how the lender's expected return varies with the parameter values of the problem, r, R, λ.

Example:

Suppose that the borrower has a quadratic utility function, $u(c(t)) = c(t) - \frac{1}{2}(c(t))^2$, and discounts future utility at a discounting factor equal to the riskless interest rate r. This violates our assumption on the utility function because the quadratic utility function has bounded marginal utility at zero consumption. This feature, however, helps to locate the credible credit limit easily. The borrower earns 1 unit of income instantaneously after becoming rich, that is, $a = 1$. We assume that the value of going bankrupt is identically 0, that is, $B(k) = 0$ for all $k \leq 0$.

The borrower's problem is written as:

$$\max_{c(t)} \int_0^T e^{-(r+\lambda)t}[c(t) - \frac{1}{2}(c(t))^2 + \lambda\overline{V}(k(t))]dt \tag{4.9}$$

$$s.t. \begin{cases} \dot{k}(t) = Rk(t) - c(t), \\ k(t) \geq -\overline{A},\ k(0) = 0, \\ c(t) \geq 0 \end{cases}$$

Once the credible credit limit is given, we can solve for the borrower's consumption path together with the time of bankruptcy, T. Notice that the maximum credit limit is $\frac{1}{r}$. If the maximum credit limit is locally profitable, it is the credible credit limit. Hence it is necessary to check that the maximum credit limit is locally profitable which holds true for all parameter values in the example.

After solving for the consumption path, we can compute the creditor's expected return from lending up to the credible credit limit. We fix the riskless rate at .003 and compute the expected return for various combinations of the parameter values of R and λ. The riskless rate of .003 is approximately the monthly interest rate when the yearly rate is 3.6%. It should be noted, however, that only the relative magnitude of the riskless rate in comparison with other parameters, R and λ, is important.

Table 4.1 shows the lender's expected return for various parameter values. It is checked that for all the parameter values, the maximum credit limit is locally

$P(0,\frac{1}{r})$ $r = 3.6\%$		$\frac{1}{\lambda}$ (expected time to be rich) 12 yrs	10 yrs	8 yrs
R	4.8%	-6.414	-2.107	-.043
	6.6%	-4.318	.202	2.373
	8.4%	-2.790	1.909	4.223

Table 4.1: Lender's Expected Return

profitable so that it is the credible credit limit. In the table, the higher R and λ, the bigger the expected return. The cases with positive expected return correspond to Figure 1 and the ones with negative expected return to Figure 2, respectively.

On the other hand, there exists a λ sufficiently low for which raising R does not make $P(0,\overline{A})$ positive. As will be shown later, there is a credit rationing if $P(0,\overline{A}) < 0$. Hence for λ sufficiently low, the credit rationing may not disappear even if the lender raises the risky interest rate R.

We first characterize the subgame perfect equilibrium for the case $P(0,\overline{A}) \geq 0$. Lemma 4 shows that $P(k_0,\overline{A})$ as drawn in Figure 1 holds true in this case.

Lemma 4 *Given any $A \geq 0$, if $P(0,A) \geq 0$, then $P(k_0,A) \geq 0$, for all $k_0 \in [-A,0]$.*

Lemma 4 implies that if $P(0,\overline{A}) \geq 0$, the expected return to the lender is positive throughout even if the borrower consumes subject to the credible credit limit. Therefore the lender always extends the loan up to the credible credit limit.

Proposition 1 *If $P(0,\overline{A}) \geq 0$, $(A^*;c^*(t))$ is the subgame perfect equilibrium in pure strategy, if $A^* = (A_1^*, A_2^*, \ldots)$ is any increasing sequence such that*

$$\sup_n\{A_n^*\} = \overline{A},$$

and strictly increasing up to $\overline{A}$, and $c^*(t) = (c_1^*(t), c_2^*(t), \ldots)$ is the solution to the problem $V(0, \overline{A})$ satisfying

$$\int_{T_{n-1}}^{T_n} e^{-Rt} c_n(t)dt = e^{-R(T_n - T_{n-1})} A_n - A_{n-1}.$$

Moreover, all the subgame perfect equilibria support a unique outcome.

In Proposition 1, the subgame perfect equilibrium results in a unique equilibrium outcome if the lender has a positive expected return from the credible credit limit. The only credit limit taken into account by the borrower is the credible credit limit. Therefore the lender cannot affect the consumption path by setting any sequence of credit limits and the optimal consumption path is uniquely determined. In this case the lender's advantage as the "first mover" in setting the credit limit totally disappears because of his changing incentive.

In the equilibrium, there is no less borrowing than in the case where the lender can directly control the consumption path. It follows that the lender's time inconsistency problem does not introduce welfare distortion but affects only the distribution of the payoffs between the players.

Although the equilibrium outcome of the game is unique in this case, there is a continuum of equilibrium strategies by the lender in the following sense: The lender can set any temporary credit limit less than the credible credit limit eventually renewed to the credible credit limit and this doesn't affect the equilibrium play of the borrower.

An additional result is presented without proof because it follows immediately from Proposition 1.

Corollary 1 *Given any outstanding debt, $-\overline{A} \leq k_0 \leq 0$, if $P(k_0, \overline{A}) \geq 0$, $(A^*; c^*(t))$ is the subgame perfect equilibrium in pure strategy of the game starting with the* *initial debt k_0, if $A^* = (A_1^*, A_2^*, \ldots)$ is any increasing sequence such that*

$$\sup_n \{A_n^*\} = \overline{A},$$

and strictly increasing up to $\overline{A}$, and $c^*(t) = (c_1^*(t), c_2^*(t), \ldots)$ is the solution to the problem $V(k_0, \overline{A})$ satisfying

$$\int_{T_{n-1}}^{T_n} e^{-Rt} c_n(t) dt = e^{-R(T_n - T_{n-1})} A_n - A_{n-1}.$$

Moreover, all the subgame perfect equilibria support a unique outcome.

Next we consider the case in which the lender stops lending before the credible credit limit because the credible credit limit does not yield a positive expected return to the lender. Because the lender should stop lending before the credible credit limit, we need an alternative point where we can start the backward induction. Such an alternative stopping point is derived in the following way.

In Figure 2, $P(k_0, \overline{A})$ crosses zero downward only once. At the debt level where $P(k_0, \overline{A}) = 0$, the lender has zero expected return from lending up to the credible credit limit. Therefore the lender is indifferent between the renewal of the credit limit and its refusal. In particular, the lender can stop lending at such a debt level so that the debt level plays the role of the credible credit limit.

Lemma 5 and Lemma 6 together with Definition 6 show the existence of such an alternative credible credit limit.

Lemma 5 *If $P(0, A) < 0$, and $P(k_0, A)$ increasing at $k_0 = -A$, there exists a unique $\overline{k} \in (-A, 0)$ such that $P(\overline{k}, A) = 0$.*

Lemma 5 shows that $P(k_0, \overline{A})$ as in Figure 2 hold true when $P(0, \overline{A}) < 0$. However, the lender may still have negative expected return from the loan up to $\overline{k}$ in Lemma 5 so that he should stop lending before $\overline{k}$. Lemma 6 shows that there is an alternative credible credit limit with the same property as $\bar{k}$ in Lemma 5 which yields the lender a positive expected return. This credit limit is computed using the sequence defined in Definition 6: the sequence $\overline{A}_i$ represents the debt levels at which the lender is indifferent between the renewal and the refusal of additional loan.

Definition 6 *Define* $\overline{A}_0 = \overline{A}$, *and recursively* $\overline{A}_i \in [0, \overline{A}_{i-1})$ *by* $P(-\overline{A}_i, \overline{A}_{i-1}) = 0$ *for* $i = 1, 2, \ldots$.

Lemma 6 *If* $P(0, \overline{A}) < 0$, *there exists a finite integer,* $i \geq 1$, *such that* $P(0, \overline{A}_i) \geq 0$.

The result of Lemma 6 can be visualized as in Figure 3. The sequence $\overline{A}_i$ in Figure 3 represents the debt level at which the lender is indifferent between the renewal and the refusal of additional loan. In the proof of Proposition 2 and Proposition 3, we use the fact that once the debt level hits $-\overline{A}_i$, the lender is indifferent between the refusal and renewal of the loan up to $-\overline{A}_{i-1}$ so that he can stop lending or use mixed strategy. Proposition 2 characterizes the subgame perfect equilibrium where the lender stops lending when he is indifferent between the renewal and the refusal of a loan while Proposition 3 takes the case of mixed strategy.

Proposition 2 *If* $P(0, \overline{A}) < 0$, *there exists a subgame perfect equilibrium with positive borrowing less than the credible credit limit in which the lender always refuses to grant a loan with zero expected return.*

Proposition 2 uses $\overline{A}_i$ in Figure 3 as the alternative credible credit limit at which the lender can stop lending. Hence the proof proceeds very much like in Proposition 1 with $\overline{A}_i$ replacing $\overline{A}$. Also the characterization is similar in that the equilibrium strategy satisfies the Markov property and uses no mixed strategy off the equilibrium path as well as along the equilibrium path; it yields the only equilibrium outcome supported by subgame perfect equilibrium in pure strategy everywhere under the condition that $P(0, \overline{A}) < 0$.

The subgame perfect equilibrium in Proposition 2 implies the presence of credit rationing because the equilibrium borrowing is less than the credible credit limit. The credit rationing in Proposition 2 is different from that in Stiglitz and Weiss(1981) where the borrower is either granted the total amount of the loan which he wants or rejected totally.

On the other hand, it seems intuitively appealing that the lender may not grant any loan initially in the case $P(0, \overline{A}) < 0$ for the following reason. If the borrower is not sure that the lender stops lending when the lender is indifferent between the renewal and the refusal of a loan with zero expected return, the borrower may consume subject to the credible credit limit leaving the lender a negative expected return. Indeed the intuition is proved to be true in the next proposition.

Proposition 3 *If $P(0, \overline{A}) < 0$, there exists a subgame perfect equilibrium with no positive borrowing in which off the equilibrium path the lender randomizes between the renewal and refusal of a loan with zero expected return.*

In the proof of Proposition 3 we need a punishment after a deviation of the lender; although no borrowing and lending is obviously a Nash equilibrium, without a punishment scheme the deviation by the lender may make the lender better off so that it is not a subgame perfect equilibrium. The punishment of the lender after a deviation requires enough probability of loss to the lender by using the mixed strategy. The mixed strategy is necessary in the punishment because the deterministic cut-off in the future may justify the current renewal of the credit limit. Notice that the mixed strategy is used only after a deviation. Hence it is never observed in the equilibrium play of the game.

The punishment using the mixed strategy is sketched as follows. The lender can use $\overline{A}_i$ in Figure 3 as the points at which he randomizes between the renewal and the refusal of additional loan. In the randomization, the lender places enough probability on the refusal so that the borrower is indifferent between consuming subject to the outstanding credit limit and a bigger credit limit and thus the borrower may randomize also. If the borrower's randomization yields a negative expected return to the lender, the lender becomes worse off after the deviation. To summarize, the uncertain future evolution of the game which is likely to yield a negative expected return prevents the lender from deviating to set a positive credit limit.

The subgame perfect equilibrium in Proposition 3 results in a total rejection of loan for a rather different reason from Stiglitz and Weiss(1981). It follows that even in the absence of asymmetric information in the relationship between the borrower and the lender, we may have no borrowing equilibrium.

4.4.2 Non-Uniqueness

Proposition 2 and Proposition 3 imply that there are multiple equilibria in the case $P(0, \overline{A}) < 0$. Both equilibria result in a borrowing less than the credible credit limit. The problem of multiplicity of the equilibrium is very severe here as we can see in the following proposition.

Proposition 4 *If $P(0, \overline{A}) < 0$, there exists a continuum of equilibria whose outcomes all result in the borrowing less than the credible credit limit.*

The continuity of equilibria arises because the equilibrium with no more borrowing can be attached in the subgame after a certain amount of borrowing. To have no more borrowing in the subgame after the equilibrium borrowing, the expected return of the lender from the loan up to the credible credit limit should be negative. It follows that the equilibrium borrowing is less than the credible credit limit.

4.5 Conclusion

This paper shows that borrowing constraints more severe than those assumed in the ideal situation when consumption is regulated by wealth may result from the equilibrium play of the game due to the time inconsistency problem. The borrowing constraint may be regarded as a kind of credit rationing. This type of credit rationing differs from other types usually documented; the borrower can be granted a loan smaller than he wants rather than either given a loan he wants or totally rejected. Moreover, our type of credit rationing may not disappear even if there are many lenders competing for the loan. Hence it invalidates the explanation of the quantity constraint in the credit market based on the monopoly power of the lender.

Our result dictates that borrowing constraints may exist without the informational asymmetry or the difficulty of enforcing repayment. Considering the practical procedure of individual loans where the bank verifies the information provided by the borrower and courts are available as the to enforcement mechanism, the two reasons may not be enough to explain the pervasive presence of borrowing constraints. Thus we can regard our model as complementary to other theories especially when the imposition of borrowing constraints is not explainable for other reasons.

Our model is a partial equilibrium model in which the interest rate is exogenously given. To fully analyze the problem of borrowing and lending, we need to extend the model such that the interest rate is endogenously determined. The obstacle to this extension lies in the computation of the lender's return for different interest rates. However, as we saw in the example, there exists a combination of values for r and λ which results in a borrowing less than the credible credit limit for any interest rates R. Although we couldn't derive a general condition, we conjecture that this happens in many cases.

The intuition of the paper may be applied to the following problem. In a game where the lender faces a sequence of group of borrowers starting to borrow at different points of time, the use of a mixed strategy along the equilibrium path has an interesting implication. To induce the borrowers to consume subject to an outstanding credit limit, the lender uses the mixed strategy of rejecting a proportion of the loan applications even if they are from the pool of applicants with the same characteristics. Hence there results credit rationing which treats loan applicants with the same characteristics differently.

4.6 Appendix

Proof of Lemma 2: Given $A \geq 0$, the optimal consumption path, $\overline{c}(t)$, is uniquely determined by solving the problem $V(0, A)$, and is continuous in t. The debt level, $\overline{k}(t)$ is also continuous in t and, moreover, strictly decreasing in t. Therefore the inverse of $\overline{k}(t)$, denoted as $\overline{k}^{-1}(\cdot)$, is continuous and decreasing in the debt level k. Fix a debt level $k_0 = \overline{k}(\tau)$, and equivalently the time at which the debt level reaches k_0 when consuming according to $\overline{c}(t)$ from 0 debt level, that is, $\tau = \overline{k}^{-1}(k_0)$. Because the consumption path from any initial debt level greater than 0 up to the same credit limit is identical by the principle of optimality, the lender's expected return from setting a credit limit A at the initial debt level of k_0 is written as the following:

$$
\begin{aligned}
P(k_0, A) &= \int_0^{\overline{T}} e^{-(r+\lambda)t}[\lambda(-\overline{k}(t)) - \overline{c}(t)]dt \\
&= \int_{\overline{k}^{-1}(k_0)}^{T} e^{-(r+\lambda)(t-\overline{k}^{-1}(k_0))}[\lambda(-\overline{k}(t)) - \overline{c}(t)]dt \qquad (4.10)
\end{aligned}
$$

where T is the time of bankruptcy for the problem $V(0, A)$.

Obviously $P(k_0, A)$ is continuous in the lower limit of the integration, $\overline{k}^{-1}(k_0)$. Because $\overline{k}^{-1}(k_0)$ is continuous in k_0, $P(k_0, A)$ is continuous in k_0. ∎

Proof of Lemma 3: We denote optimal solution of the borrower's problem by overline over the variable as in the proof of Lemma 2. First notice that if the integrand of $P(k_0, A)$, $-\lambda\overline{k}(t) - \overline{c}(t)$, is increasing in t everywhere, then we are done because to the left of t for which $-\lambda\overline{k}(t) - \overline{c}(t) = 0$, $P(k_0, A)$ is decreasing in k_0.

To get the lemma, we only need to guarantee that $-\lambda\overline{k}(t) - \overline{c}(t)$ is increasing at t for which $-\lambda\overline{k}(t) - \overline{c}(t) = 0$, because in that case, $-\lambda\overline{k}(t) - \overline{c}(t)$ remains positive once it becomes positive as k_0 decreases, or equivalently, $\tau = \overline{k}^{-1}(k_0)$ increases.

The Euler equation for the problem $V(k_0, A)$ is written as:

$$\begin{aligned}\frac{d}{dt}u'(c(t)) &= u''(c(t))\frac{d}{dt}c(t) \\ &= -(R-\delta-\lambda)u'(c(t)) - \lambda\overline{V}'(k(t)). \end{aligned} \tag{4.11}$$

We can write the time derivative of the function $-\lambda\overline{k}(t) - \overline{c}(t)$ as follows:

$$\begin{aligned}&\frac{d}{dt}[-\lambda\overline{k}(t) - \overline{c}(t)] \\ &= -\lambda(R\overline{k}(t) - \overline{c}(t)) + \frac{1}{u''(\overline{c}(t))}[(R-\delta-\lambda)u'(\overline{c}(t)) + \lambda\overline{V}'(\overline{k}(t))]\end{aligned} \tag{4.12}$$

using $\dot{k}(t) = Rk(t) - c(t)$, and the above Euler condition (4.12).

Since for all k_0, $u'(\overline{c}(t)) \geq \overline{V}'(\overline{k}(t))$(Lemma 1(b)), we evaluate the above time derivative at t such that $-\lambda\overline{k}(t) - \overline{c}(t) = 0$:

$$\begin{aligned}&\frac{d}{dt}[-\lambda\overline{k}(t) - \overline{c}(t)]_{-\lambda\overline{k}(t)-\overline{c}(t)=0} \\ &\geq (R+\lambda)\overline{c}(t) + \frac{1}{u''(\overline{c}(t))}[(R-\delta-\lambda)u'(\overline{c}(t)) + \lambda u'(\overline{c}(t))] \\ &= (R+\lambda)\overline{c}(t) + \frac{1}{u''(\overline{c}(t))}(R-\delta)u'(\overline{c}(t)).\end{aligned} \tag{4.13}$$

If the last line of equation (4.13) is greater than 0, $-\lambda\overline{k}(t) - \overline{c}(t)$ is increasing at t for which $-\lambda\overline{k}(t) - \overline{c}(t) = 0$. Therefore if

$$(R+\lambda)\overline{c}(t) \geq -\frac{1}{u''(\overline{c}(t))}(R-\delta)u'(\overline{c}(t)),$$

or equivalently,

$$-\frac{cu''(c)}{u'(c)} \geq \frac{R-\delta}{R+\lambda},$$

then $-\lambda\overline{k}(t) - \overline{c}(t)$ is increasing at t for which $-\lambda\overline{k}(t) - \overline{c}(t) = 0$.

Therefore we have $\hat{t}$ such that $-\lambda\overline{k}(t) - \overline{c}(t) \geq 0$ for all $t \geq \hat{t}$ and vice versa. Because $\overline{k}^{-1}(k_0)$ is a decreasing function of k_0, $-\lambda\overline{k}(t) - \overline{c}(t)$ is always negative as k_0 increases from $k(\hat{t})$. It follows that

$$P(k_0, A) = \int_{\overline{k}^{-1}(k_0)}^{T} e^{-(r+\lambda)(t-\overline{k}^{-1}(k_0))}[\lambda(-\overline{k}(t)) - \overline{c}(t)]dt$$

is decreasing for all $k_0 \geq k(\hat{t})$, and it is decreasing when $P(k_0, A) \leq 0$, *a fortiori.*

∎

Proof of Lemma 4: Aiming at contradiction, suppose that there exists $\overline{k} \in [-A, 0]$ such that $P(k_0, A) < 0$. Then in the neighborhood to the right of $\overline{k}$, $P(k_0, A)$ is decreasing in k_0 by Lemma 3, and $P(k_0, A) < 0$ for all $k_0 > \overline{k}$. In particular, $P(0, A) < 0$ which is a contradiction to our hypothesis. ∎

Proof of Proposition 1: Given A^* as in the proposition, it is obvious that $c^*(t)$ is the best response of the borrower. Note that A^* is a strictly increasing sequence up to $\overline{A}$ because the game ends if $A_{n-1} = A_n$.

To show that A^* is the best response to $c^*(t)$, notice that if $\sup_n\{A_n^*\} < \overline{A}$, there exists another strategy, A', such that

$$\sup_n\{A_n'\} \in (\sup_n\{A_n^*\}, \overline{A}],$$

and

$$P_n(A^*; c^*(t)) \leq P_n(A',; c^*(t))$$

for all n, with strict inequality for all $n \leq N$ for some N.

This is possible because there exits an additional credit limit with positive expected return for any credit limit less than the credible credit limit. Therefore contradiction.

Next suppose that $\sup_n\{A_n^*\} > \overline{A}$. Without loss of generality, assume $A_1^* = \overline{A}$, and $A_2^* > \overline{A}$. Then

$$P_2(A_2^*, A_{-2}^*; c^*(t)) < P_2(A_2 = \overline{A}, A_{-2}^*; c^*(t)) = 0.$$

Therefore A^* is not a best response. This proves that $(A^*, c^*(t))$ is a Nash equilibrium.

To prove that $(A^*, c^*(t))$ is a subgame perfect equilibrium, consider any history A_{n-1} at $t = T_{n-1}$. By Lemma 4, $(A^*|A_{n-1}, c^*(t)|A_n)$ is a Nash equilibrium for all $\Gamma_n^l|A_{n-1}$, $n = 1, 2, \ldots$. It is trivial to show that after any history A_n at $t = T_{n-1}$, $(A^*|A_n, c^*(t)|A_n)$ is a Nash equilibrium for all $\Gamma_n^b|A_n$, $n = 1, 2, \ldots$. This completes the proof that $(A^*, c^*(t))$ is a subgame perfect equilibrium in pure strategy.

The proof of the uniqueness of the equilibrium outcome is straightforward because $\overline{A}$ is unique by definition. ∎

Proof of Lemma 5: Because $P(k_0, A)$ is continuous in k_0 by Lemma 2 and $P(k_0, A)$ is increasing at $k_0 = -A$, we have $\overline{k} \in (-A, 0)$ at which $P(\overline{k}, A) = 0$ by the intermediate value theorem.

The uniqueness follows because by Lemma 3, $P(k_0, A)$ remains negative for all $k_0 \in [\overline{k}, 0]$ once it becomes negative at $\overline{k}$. ∎

Proof of Lemma 6: The sequence $\{\overline{A}_i\}$ exists by Lemma 5 and is strictly decreasing by Definition 6. The difference between any two consecutive $\overline{A}_i$'s is bounded away from zero. Denote the minimum difference by d. Then there exists an integer N such that $\overline{A} - nd \leq 0$ for all $n \geq N$, i.e., the sequence $\{\overline{A}_i\}$ eventually reaches 0. Because there exists a credit limit which gives the lender a positive expected return at $k_0 = 0$, we have an integer, $1 \leq i \leq N$, such that $P(0, \overline{A}_i) \geq 0$.

∎

Proof of Proposition 2: Suppose $P(0, \overline{A}_1) \geq 0$, followed by a subgame with zero expected return to the lender. By Corollary 1, after a history $A_n > \overline{A}_1$, the lender's unique subgame perfect equilibrium strategy is renewing the credit limit up to $\overline{A}$. On the other hand, after a history $A_n = \overline{A}_1$, the creditor refuses any additional loan by hypothesis because $P(-\overline{A}_1, \overline{A}) = 0$.

Consider the following strategy pair $(A^*, c^*(t))$:

$$\max\{A_n^*\} = \overline{A}_1$$

and $c^*(t)$ solves problem $V(0, \overline{A}_1)$.

It is obvious that the given strategy pair is a Nash equilibrium. It suffices to prescribe the Nash equilibrium after any deviation from the given strategy.

After a deviation by the lender such that $A_n > \overline{A}_1$, the borrower consumes subject to the credible credit limit $\overline{A}$ and the lender extends the credit limit up to $\overline{A}$. This constitutes a Nash equilibrium in the subgames $\Gamma_n^b | A_n$ and $\Gamma_{n+1}^l | A_n$. But the deviation makes the lender worse off because for $A_{n-1} < \overline{A}_1$, $P(-A_{n-1}, \overline{A}) < 0$ and consequently $P_n(A_n, A_{-n}^*; c^*(t)) < 0$. This shows that $(A^*, c^*(t))$ is a subgame perfect equilibrium for the case $P(0, \overline{A}_1) \geq 0$.

For the case where the credit limit with positive expected return is reached after i steps, $i \geq 2$, there is no loss of generality assuming that $P(0, \overline{A}_2) \geq 0$. After a history $A_n \in (\overline{A}_2, \overline{A}_1)$, it is a Nash equilibrium that the lender stops only at $\overline{A}_1$ because $P(-A_n, \overline{A}_1) > 0$. We can apply the same argument as above with $\overline{A}_1$ replacing the credible credit limit $\overline{A}$. It is routine to check that $(A^*, c^*(t))$ is a subgame perfect equilibrium if

$$\max\{A_n^*\} = \overline{A}_2$$

and $c^*(t)$ solves problem $V(0, \overline{A}_2)$.

Finally by Lemma 6 there exists $\overline{A}_i$ such that $P(0, \overline{A}_i) \geq 0$. Therefore there exists a subgame perfect equilibrium with a positive borrowing. ∎

Proof of Proposition 3: We construct such an equilibrium. The equilibrium uses mixed strategies after a deviation by the lender. We have to relax the assumption that the strategy satisfies the Markov property because the mixed strategy depends on the debt level at which the deviation takes place.

Step 1:

As in Proposition 2, we first suppose that $P(0, \overline{A}_1) \geq 0$, followed by a subgame with zero expected return to the creditor.

Consider $(A^*, c^*(t))$such that $A_1^* = 0$ and the sequence $A_n^*, n \geq 2$, is a strictly increasing sequence up to $\overline{A}$ and $c^*(t)$ solves problem $V(0, \overline{A})$. Such a strategy pair is a Nash equilibrium because given his own strategy for $n \geq 2$ and the borrower's strategy, the lender cannot initially deviate with positive expected return and given the lender's strategy the borrower's deviation does not affect the play of the game.

The above strategy characterizes the equilibrium only along the equilibrium path; For the above strategy to be a subgame perfect equilibrium, we need to check the Nash equilibrium in any subgame after deviation. There are three types of deviations possible, $A_1 > \overline{A}_1$, $A_1 = \overline{A}_1$, and $A_1 < \overline{A}_1$. We analyze the play after each type of deviation.

Case 1:

Suppose that the lender deviates to $A_1 > \overline{A}_1$. After the deviation $c^*(t)$ as above is the borrower's best response because $P(-A_1, \overline{A}) > 0$ so that by Corollary 1 $\max\{A_n\} = \overline{A}$ is the unique Nash equilibrium in the subgame $\Gamma_1^b | A_1$. The deviation makes the lender worse off because $P(0, \overline{A}) < 0$ and $P_1(A_{-1}, A_{-1}^*; c^*(t)) < 0 = P_1(A^*; c^*(t))$.

Case 2:

Consider a deviation by the lender such that $A_1 = \overline{A}_1$. After the deviation the lender is indifferent between the renewal and the refusal of any additional loan beyond it because $P(-\overline{A}_1, \overline{A}) = 0$. The lender randomizes so that the borrower is indifferent between consuming subject to $\overline{A}_1$ and $\overline{A}$, i.e.,

$$\pi V(0, \overline{A}_1) = (1-\pi) V(0, \overline{A}), \tag{4.14}$$

where π is the probability of refusal at $\overline{A}_1$. Given the mixed strategy of the lender, the borrower randomizes between consuming subject to $\overline{A}_1$ and $\overline{A}$ so that the lender is not better off after the deviation, i.e.,

$$\mu P(0, \overline{A}_1) + (1-\mu) P(0, \overline{A}) \leq 0, \tag{4.15}$$

where μ is the probability that the borrower consumes subject to $\overline{A}_1$.
Because $P(0, \overline{A}_1) \geq 0$ and $P(0, \overline{A}) < 0$, we can make the lender worse off by choosing small μ. The strategy after the deviation constitutes a Nash equilibrium in the subgame $\Gamma_1^b | A_1$ and the lender becomes worse off by the deviation.

Case 3:

Finally consider a deviation by the lender such that $A_1 < \overline{A}_1$. We need to characterize a Nash equilibrium in the subgame $\Gamma_1^b | A_1$, and show that deviation gives the lender smaller expected return than the equilibrium play.

Notice that if the credit limit is extended further to $\overline{A}_1$, the lender randomizes between the renewal and the refusal so that the borrower is indifferent,

$$\pi V(-A_1, \overline{A}_1) = (1-\pi) V(-A_1, \overline{A}). \tag{4.16}$$

Suppose that when the borrowing reaches the given credit limit, i.e., $k(T_1) = -A_1$, the lender randomizes between the renewal to $A_2 = \overline{A}_1$ and refusal at A_1. To induce the randomization, when $k(T_1) = -A_1$, the borrower should randomize so that the lender is indifferent between the two strategies, i.e., the lender's expected payoff from the renewal equals to 0 which is the payoff from the refusal:

$$\mu_0 P(-A_1, \overline{A}_1) + (1-\mu_0) P(-A_1, \overline{A}) = 0 \tag{4.17}$$

where μ_0 is the probability that the borrower consumes subject to $\overline{A}_1$ and $(1-\mu_0)$ is the one for $\overline{A}$.

Also the lender initially commits to a mixed strategy at $k(T_1) = -A_1$ so that the borrower is indifferent between consuming subject to A_1 and subject to randomized credit limit of $\overline{A}_1$ and $\overline{A}$.

$$\pi_1 V(0, A_1) = (1-\pi_1)[\pi_0 V(0, \overline{A}_1) + (1-\pi_0) V(0, \overline{A})] \tag{4.18}$$

where π_1 is the probability that the lender cuts off at A_1 and π_0 is the probability that he cuts off at $\overline{A}_1$.

The above strategy pair is a Nash equilibrium in mixed strategy in the subgame $\Gamma_1^b | A_1$.

To support the strategy $A_1^* = 0$ by subgame perfect equilibrium, we need to show that the lender gets worse off by the deviation. If the borrower randomizes at $k(T_0) = 0$ such that

$$\mu_1 P(0, A_1) + (1-\mu_1)[\mu_0 P(0, \overline{A}_1) + (1-\mu_0) P(0, \overline{A})] \leq 0 \tag{4.19}$$

where μ_1 is the probability to consume subject to A_1, the lender gets smaller expected return from deviation and $(A^*, c^*(t))$ is a subgame perfect equilibrium.

However it is possible that there is no μ_0 satisfying equation (4.17) and equation(4.19) simultaneously, i.e., any μ_0 satisfying

$$\mu_0 P(-A_1, \overline{A}_1) + (1-\mu_0) P(-A_1, \overline{A}) = 0, \tag{4.20}$$

gives strictly positive expected return to the lender such that

$$\mu_1 P(0, A_1) + (1-\mu_1)[\mu_0 P(0, \overline{A}_1) + (1-\mu_0) P(0, \overline{A})] > 0. \tag{4.21}$$

because equation (4.20) does not necessarily imply that the term in the square brackets in equation (4.21) is negative. If this term is positive, equation (4.21) holds for any μ_1 because $P(0, A_1) \geq 0$. Therefore given the mixed strategy of the borrower the lender gets better off by deviation. To prevent profitable deviation by the lender in this case, the borrower randomizes at $k(t) = \overline{A}_1$ so that

$$\mu_0 P(-A_1, \overline{A}_1) + (1 - \mu_0) P(-A_1, \overline{A}) \geq 0, \tag{4.22}$$

and

$$\mu_0 P(0, \overline{A}_1) + (1 - \mu_0) P(0, \overline{A}) \leq 0. \tag{4.23}$$

This is identical to the deviation seen in step 2 above.

Any additional deviation in the subgame after the initial deviation belongs to one of the above three cases and can be handled accordingly. It completes the proof of the theorem when $P(0, \overline{A}_1) \geq 0$.

Step 2:

When $P(0, \overline{A}_i) \geq 0$, for $i \geq 2$, without loss of generality we can assume that $P(0, \overline{A}_2) \geq 0$.

Before we consider the play after a deviation, we provide a result which narrows the range of deviations we have to examine. It can be easily shown that if $P(0, \overline{A}_1) < 0$, then $P(0, A_1) < 0$ for any $A_1 \geq \overline{A}_1$. Hence any deviation by the lender such that $A_1 \geq \overline{A}_1$ definitely makes the lender worse off because even if the borrower consumes subject to A_1, the creditor gets negative expected return. From the observation we can confine ourselves to the deviation A_1 such that $P(0, A_1) \geq 0$ and $A_1 < \overline{A}_1$. Furthermore, if it is a Nash equilibrium in the subgame after the deviation that the lender renews up to $\overline{A}_1$, the lender definitely gets worse off by the deviation. With these facts in mind, we can proceed as in Step 1.

Case 1:

Consider a deviation $A_1 \in (\overline{A}_2, \overline{A}_1)$ such that $P(0, A_1) \geq 0$. It suffices to show that in the subgame $\Gamma_1^b | A_1$, the lender renews the credit limit at least up to $\overline{A}_1$. We can achieve this by letting the borrower randomize at $k(t) = A_1$ so that

$$\mu P(-A_1, \overline{A}_1) + (1 - \mu) P(-A_1, \overline{A}) > 0. \tag{4.24}$$

Case 2:

If the lender deviates to $A_1 = \overline{A}_2$, he is indifferent to renewing the credit limit up to $\overline{A}_1$ and refusing at $\overline{A}_2$ because $P(-\overline{A}_2, \overline{A}_1) = 0$. Hence we apply the same argument as Case 2 of Step 1 to show that from the equilibrium play the lender gets worse off after the deviation.

Case 3:

Suppose that the lender deviates to $A_1 \in (0, \overline{A}_2)$. Let the lender cut off at $\overline{A}_1$ so that $\overline{A}_1$ plays the role of the credible credit limit. Then the case is identical to Case 3 of Step 1 and we are done. ∎

Proof of Proposition 4: If $P(0, \overline{A}_1) \geq 0$, consider the lender's strategy A^* such that $\max\{A_n^*\} = \hat{A} \leq \overline{A}_1$. In the subgame $\Gamma_n^l | A_n^*$, where $A_n^* = \hat{A}$, we have a subgame perfect equilibrium with no more borrowing by Proposition 3. Hence we have the desired equilibrium.

If $P(0, \overline{A}_2) \geq 0$, consider the lender's strategy A^* such that $\max\{A_n^*\} = \hat{A}$ where $P(0, \hat{A}) \geq 0$. Notice that in this case $\hat{A}$ may exceed $\overline{A}_2$ but is less than $\overline{A}_1$. By attaching a subgame perfect equilibrium with no more borrowing in the subgame $\Gamma_n^l | A_n^*$, we have a positive borrowing subgame perfect equilibrium. ∎

Figure 4.1: Lender's Expected Return: Case 1

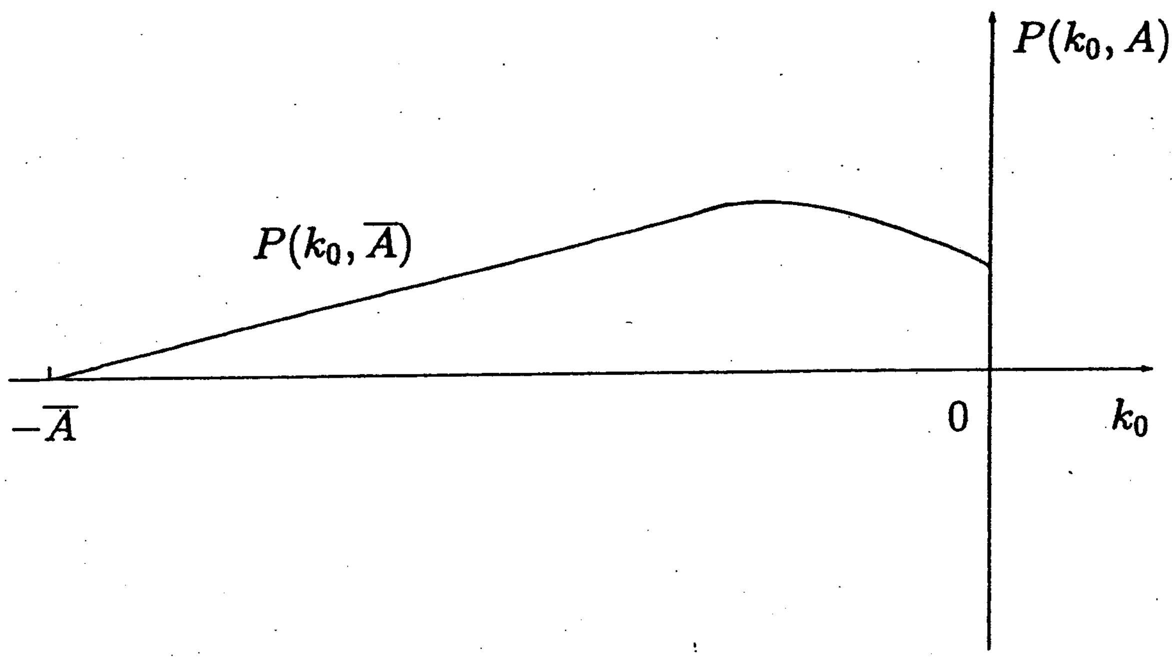


Figure 4.2: Lender's Expected Return: Case 2

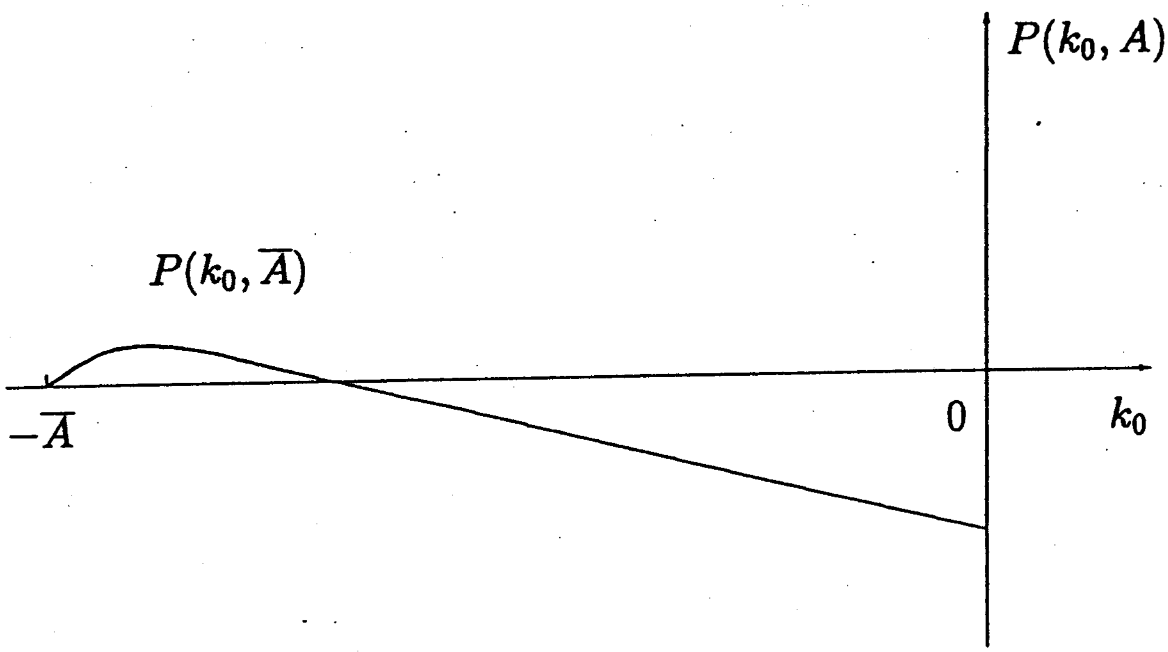
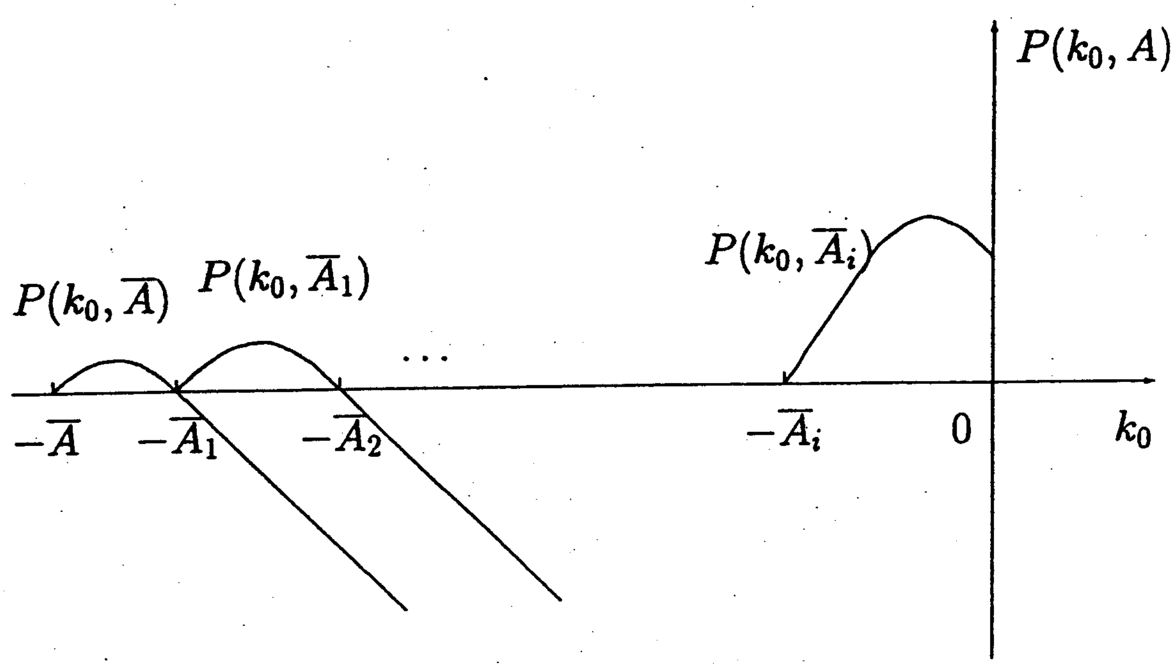


Figure 4.3: Alternative Credible Credit Limits



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